Optimal Investment Under Uncertainty Regarding Income Subsidies

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Abstract

This paper studies optimal investment decision under uncertainty regarding agricultural income subsidies. The approach is based on stochastic programming. Investment decision is modelled as a Markov decision process. The cost of imperfect information can be estimated as the difference between the profitability of investment under stable income subsidies and under uncertain subsidies. risk. Assuming random income is stationary or non-increasing over time, there is little scope for an option value of postponing investment. A source of an option value assuming non-increasing non-stationary random income comes from the explicit modelling of a cost associated with risk (variance). Examples suggest that the optimal timing of the investment is sensitive to the modelling of risk.

1 Introduction

Common Agricultural Policy (CAP) after EU enlargement implies many uncertainties regarding future agricultural income in Northern Europe. This paper addresses the impact of uncertainty regarding income subsidies and the temporary nature of the investment programs on investment behavior in Finnish agriculture. The efficiency of the investment subsidy system can be studied applying investment options [2]. Investment options typically involve three parameters: the initial and accumulated costs, the flexibility in timing the investment and the uncertainty regarding the future rewards. [2] studies the optimal investment decision as function of these parameters, applying a Markov decision process (MDP) that is defined in continuous time (Ito process) and with a continuous state space. However, a continuous time MDP can not be solved efficiently by standard dynamic programming methods. This paper applies a discrete time Markov model to study optimal investment under uncertain income subsidies and temporary investment programs.

Section 2 summarizes the underlying deterministic model of the optimal timing of the investment decision based on [2]. Section 3 presents two Markov models of optimal investment under uncertainty regarding future income subsidies, applying a stochastic programming approach. Both models are based on a Markov model, allowing for policy/time-dependent transition probabilities (e.g. due to temporary investment subsidies). In the first model, the state of the system is defined in terms of the value of investment. Two cases are considered: at the time when investment is made, the state of the system is assumed to be either fully known or known in terms of a probability distribution. The cost of imperfect information can be estimated as the difference between the profitability of investment under certain state (corresponding to stable income subsidies) and under uncertain state (assuming uncertain subsidies). In the second model, the value of investment at the time investment is made is unobservable [1].

Recently, [12] has applied a discrete time MDP model to study optimal farmland investment assuming a standard price subsidy or a decoupled direct payment. [12] is based on assuming a risk-neutral decision maker maximizes the firm's expected net worth. In [12] the investment results are not very sensitive to the variability in the income. This is due to the assumed stationarity of the revenues. However, as emphasized in [2], in general both the growth in the value of investment and uncertainty (as modelled by the variability in income) affect the optimal timing of the investment. To capture the effect of the variability in income on investment in a discrete time model, the MDP model is extended in section 4 to explicitly account for risk, following [9].

When a firm makes an irreversible investment, it gives up the possibility of waiting for new information to arrive that might affect the desirability or timing of the expenditure [2]. The option to postpone the investment gives an option value to the investment: the difference between the net value of

[1] In addition to uncertain subsidies, there are of course other reasons to a stochastic return on investment, including uncertain costs and demand.

the investment with optimal timing and the net value obtained when the investment is made immediately. This value is the larger the more information becomes available by waiting. In this paper future income is assumed to be non-stationary: the distribution of the random income subsidies is assumed to be time-varying, with a non-increasing expected value. This implies, whether the state is fully observable or not, that for a risk-neutral decision maker there is little scope for an option value of postponing investment. Numerical results are discussed in section 5. The only source of an option value assuming non-increasing subsidies comes from the explicit modelling of a cost associated with risk (variance). The main results from the simulations are as follows:

- The cost of uncertainty can be high. Lack of complete information about subsidies causes inefficiencies, cf. [8]. This implies is a trade-off between the stability and continuity of a subsidy program and the expected subsidy level;

- Examples of MDPs with risk suggest that the optimal investment decision can be sensitive to risk.

2 Optimal Stopping Model: Deterministic Case

The concept of an investment option is introduced in what follows, first assuming a deterministic model. Consider the investment decision of a representative firm. For simplicity, assume time t is discretized $t = 1, ..., T$. Investment cost or return when postponed by one time unit is discounted by the factor $b \in (0, 1]$ where a lower b indicates a higher preference for immediate return.

Let r_t denote the rate of return per unit investment at time t. At time $t = 0$ the future rate of return, $r_t,\ t > 0$ is a random variable. For simplicity, consider the timing of the investment as the decision variable. The investment decision at any time period is binary: to invest or postpone investment; denoting by I the total available budget for the investment, the decision I_t at t satisfies

$$I_t \in \{0, I\}\ \forall t. \tag{1}$$

The aggregate budget constraint requires:

$$\sum_t I_t \leq I. \tag{2}$$

I.e., to begin with it is assumed that the total available amount for investment can be spent at any time t; later this simplification is removed by introducing period-specific budget constraints. Letting E denote the expectation operator, the dynamic optimization problem can be written in discrete-time as:

$$\max\ E[\sum_{k=0}^{T}\sum_{t=k}^{T} b^t I_k r_t - b^k I_k] \tag{3}$$

subject to (1)-(2).

Optimal Timing of Investment

Suppose first that the yield is deterministic over time $E(r_t) = r_t$ at $t = 1, 2, ..., T$. Continuous time interest rate v corresponding to discount factor b implies a discrete time interest rate ρ defined by

$$\frac{1}{(\rho+1)^t} = b^t = e^{-vt},$$

i.e. $v = -\log(b)$. Following [2], in a continuous time model, define the value of the investment opportunity when investing at time T^* as

$$F(V) = (Ve^{\alpha T^*} - I)e^{-vT^*} \tag{4}$$

where $\alpha \geq 0$ denotes a growth parameter. If $\alpha = 0$, it is optimal to invest immediately, provided $V \geq I$. Assuming $0 < \alpha < v$, the optimal time for the investment T^* in a continuous time model satisfies [2]:

$$T^* = \max\{\frac{1}{\alpha}\log(\frac{vI}{(v-\alpha)V}), 0\}. \tag{5}$$

Option Value

The option value of the investment opportunity is defined in [2] as the difference

$$\omega = F(V(T^*)) - (V - I). \tag{6}$$

where $F(V)$ is defined in (4) and V is equal to the value of investment made at $T = 0$. Let

$$V^* = \frac{v}{v - \alpha} I. \tag{7}$$

If $V > V^*$ where V^* is defined above, $\omega = 0$. Assuming $V \leq V^*$,

$$\omega = I(\frac{\alpha}{v - \alpha}[\frac{1 + r}{1 - b}\frac{(v - \alpha)}{v}]^{v/\alpha} - (\frac{1 + r}{1 - b} - 1)) \text{ for } V \leq V^*.$$

The discrete time model with value given by (3) is different from the above continuous time model: in particular, the discrete time model is based on assuming that if investment is made at time T^*, the value obtained depends on the income at $t = T^*, .., T$:

$$\sum_{t=T^*}^{T} b^t r_t I_{T^*}. \tag{8}$$

By (8) the value in (4) corresponds to the value (3) in the discrete time model only if

$$b^{T^*} V e^{\alpha T^*} = I \sum_{t=T^*}^{T} b^t r_t. \tag{9}$$

Let $y_t = \frac{r_t I}{1-b}$. Assuming r_t is constant for $t = T^*, .., T$, the discrete time value in (8) is

$$b^{T^*} y_{T^*} \equiv b^{T^*} \frac{r_{T^*} I}{1 - b}. \tag{10}$$

The discrete time value in (10) corresponds to the continuous time value in left-hand-side of (9) when letting

$$\frac{r_t I}{1 - b} = V e^{\alpha t}. \tag{11}$$

If $V = I$ denotes the value of investment at $t = 0$, (11) requires:

$$r_t/(1 - b) = e^{\alpha t}.$$

Thus, a positive growth rate $\alpha > 0$ is equivalent to the requirement that $\{r_t\}$ is an increasing sequence in time. There is, however, little reason to expect $\{r_t\}$ to be increasing in time (see following example). To find an option value in agricultural investments, it becomes necessary to explicitly consider risk.

Table 1: Return on investment (ROI %) in milk production (2007 -10 percent means 2007 ROI (including subsidy) when producer price decreases by 10 % from 2003 and investment subsidy increases by 20 % or 50 % depending on production unit

	ROI %	ROI %
	herd size 60	herd size 130
2003	24	30
2007 -10 %	10	16
2007 -12 %	7	14
2007 -15 %	4	11
2007 -17 %	2	8
2007 -20 %	0	5

Example

Table 1 summarizes the expected profitability of investment in milk production, based on [11]. Assuming the investment subsidy grows by 20 percent and assuming the income subsidy decreases by 15 percent from 2003 level by 2007, the profitability of a livestock-place is 11 percent 2007 (assuming herd size 130). Setting $T^* = 0$ in (5), it is optimal to invest immediately whenever $V \geq V^*$ where V^* is given in (7). For the continuous time problem to make sense it is necessary to assume that $\alpha < v$; otherwise, waiting longer would always be a better policy [2]. Postponing investment is not optimal when $\alpha = 0$.

For details regarding Table 1, see [11]. Section 5 returns to the example in Table 1.

3 Optimal Investment under Uncertainty

In general, the optimal timing of investment depends on both the growth rate α and the variability of the return r_t over time. Assume in what follows that the random variation in r_t is due to randomly varying income subsidies. Suppose for simplicity the uncertainty regarding the future income subsidies can be modelled as a Markov model. The Markov decision process (Ito process) in [2] is in continuous time and has a continuous state space. To simplify numerical analysis, this paper focuses on discrete time models.

3.1 Markov Model

Assume the possible Markov states are defined in terms of the future rate of return. For example, in Table 1 the states of future rate of return depend on the herd size and future income subsidy. Consider a discrete time model for a given herd size. The transition probabilities between the different possible states are given in Table 2. Due to fixed term investment subsidy programs, the matrix of transition probabilities depends on the timing of the investment. Denote the matrix of transition probabilities at time t by $\mathbf{A}_t$. Let r_{it} denote the rate of return at time t when the subsidy is determined by state i. The expected return $E(r_t)$ at time t is defined as:

$$E(r_t) = \sum_i P_{it} r_{it} \tag{12}$$

where P_{it} is the probability that the rate of return is determined by state i at time t and r_{it} is the rate of return at time t is state i. The probabilities $\mathbf{P}_t = \{P_{it}\}$ associated with the different states r_{it} at time t are determined from:

$$\mathbf{P}'_t = \mathbf{P}'_0 \mathbf{A}^t_t, \tag{13}$$

where $\mathbf{P}_0$ denotes the vector of initial probabilities of the different subsidy states.

Transition Probabilities

The transition probabilities in Table 2 are defined independent of the investment policy. In general, the transition probabilities at time t can be defined as function of the investment policy at time t, e.g. accounting for a

Table 2: Transition Probabilities between Different Scenarios Regarding Producer Price Change

	0%	-10 %	-12 %	-15 %	-17 %	-20 %
0 %	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
-10 %	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}
-12 %	p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}
-15 %	p_{41}	p_{42}	p_{43}	p_{44}	p_{45}	p_{46}
-17 %	p_{51}	p_{52}	p_{53}	p_{54}	p_{55}	p_{56}
-20 %	p_{61}	p_{62}	p_{63}	p_{64}	p_{65}	p_{66}

temporary investment program. Formally, letting $S(t)$ denote the state at time t and I_t denote the investment decision at time t, the state transition probability is given as a function

$$p(S(t)|S(t-1), I_t). \tag{14}$$

A *Markov Decision Process* (MDP) is a Markov chain with the above modification, i.e. the transition probability matrix depends on the action taken in each stage. For example, investment may increase productivity [6]: this can be modelled by a MDP with a slightly more advantageous transition matrix (relatively increasing the probabilities of moving to a higher return state) whenever investment takes place. Two models are considered in what follows. In the first model in section 3.2, the state of the system (in terms of return on investment) is assumed to be known when making an investment decision. In the second model in 3.3, the state of the system is unobservable. It is not necessary to assume a constant transition matrix (continuing investment subsidy program); Section 5 discusses examples with policy/time-dependent transition-probabilities.

3.2 Model 1: Discrete Time Dynamic Program

Consider the following sequential decision problem in discrete time. At time $t = 1, .., T$ the firm makes a decision on the level of investment

$$I_t \in S = \{0, 1, 2, .., I\},\ t = 1, .., T, \tag{15}$$

where I denotes the maximum available budget for investment per time period. The budget constraint for cumulative investment thus requires:

$$\sum_t I_t \leq TI. \tag{16}$$

An observed return rate r_t at time t is generated by a transition matrix, in general depending on the investment policy. It is assumed for simplicity that when making investment decision at time t, r_t remains the same for the future periods $t + 1, .., T$. The dynamic optimization problem subject to (15)-(16) thus can be stated as

$$v(r_0) = \max \sum_{t=0}^{T} b^t E(y_t - I_t) \tag{17}$$

where $v(r_0)$ denotes the value function given initial state r_0 and where y_t is given by (cf. equation (10))

$$y_t = \frac{r_t I_t}{1 - b}.$$

Problem (17) subject to (15)-(16) can be solved recursively applying Bellman equation:

$$v(r_t) = \max_{I_t \in S} \{(y_t - I_t) + bEv(r_{t+1})\}.$$

The expected value of perfect information measures the maximum amount a decision maker would be willing to pay for complete information about the future [1].

Definition 1 *Let $f(x)$ denote the objective function to be maximized with respect to decision variable x. Let z denote a random variable. The expected value of perfect information (EVPI) can be measured as the difference [1]*

$$E[\max\ f(x, z)] - \max\ E(f(x, z)). \tag{18}$$

The first term in equation (18) corresponds to a "wait-and-see" solution and the second term to an expected value maximizing solution. A numerical example of EVPI will be given in 5.3. EVPI can be applied to measure the cost of imperfect information under randomly varying subsidies, in comparison to stable (non-random) subsidies. Applying (18) to value function v implies period-t EVPI(t) in terms of value function:

$$EVPI(t) = E(v(r_t)) - v(E(r_t)).$$

3.3 Model 2: Unobservable Value

Due to the many uncertainties regarding future agricultural subsidies, it seems realistic to assume that information becomes available only gradually. This motivated the modelling of the value of investment as non-observable. Consider now a model with an aggregate budget constraint given by:

$$\sum_t I_t \leq I.$$

The expected value of investment $I_N \in S$ at stage N can be defined as

$$E(V(N)) = \sum_{t=N+1}^{T} e^{-vt} I_N E(r_t) - e^{-vN} I_N, \tag{19}$$

where $E(r_t) = \sum_i r_{it} P_{it}$ is computed using probabilities $\{P_{it}\}$ given in (13). Here the value of investment made at time $t = N$ in (19) is given in terms of an expected value: assuming r_t is observed at the end of period t, all terms r_t affecting the value of investment made at time $t = N$ are random at the beginning of period t. This is different from the above model in section 3.2 where the value of investment at any time t only depends on the history of realizations up to time t.

The two terms in (19) correspond to the expected present value of the cost and revenue. In what follows, denote the expected revenue from making investment I_N by

$$E(R(N)) = E[\sum_{t=N+1}^{\infty} b^t I_N r_t e^{\alpha t}]. \tag{20}$$

Define the state of the system S as the amount of budget available. Denoting the next state after decision I_N by S', the state transition equation giving the resultant state at stage N is given by

$$S' = S - I_N. \tag{21}$$

The model can be solved by backward recursion. The first stage $N = 1$ thus corresponds to the last time period T. On average, the return at time $t = T$ with $N = 1$ is given by (12) with $t = T$. Let $F_N(S) = \sum_{k=1}^{N} E(V(k))$ denote the overall return starting from state S at stage N. The Bellman equation can be written as:

$$F_N(S) = \max_{I_N}\{E(R(N)) + E[F_{N-1}(S'(N, S, I_N))]\}. \tag{22}$$

4 Risk in Markov Decision Processes

The above MDP models are based on expected net income maximization and assume risk neutrality. For example, consider Model 2 above. Since the value of the investment only depends on random future values, the expected value of perfect information applying Definition 1 will be zero. If the income process is non-increasing in time, there is no option value of postponing investment. As emphasized in [2], in general both a positive growth rate $\alpha > 0$ and uncertainty (as modelled by the variability in income) can create a value for postponing investment. To take risk explicitly into account in the Markov model, consider the following modification of a MDP based on [9].

The idea that risk affects decision-making is not new in agricultural economics; stochastic programming has been applied to decision-making in agriculture under uncertainty [4]. A traditional approach can be summarized as follows (ibid.). Consider a utility function in exponential form:

$$U(x) = 1 - e^{-\beta x}, \tag{23}$$

where β is a risk-aversion parameter. The expected value of utility (23) is

$$E(U(x)) = E(x) - \frac{\beta}{2} Var(x), \tag{24}$$

corresponding to a static objective function (ibid.). A corresponding dynamic objective function accounting for risk is defined next following [9].

A Stochastic Programming Model

Consider the quadratic utility function:

$$U(x) = x - \frac{\beta}{2}x^2. \tag{25}$$

Definition 2 *The resource certainty equivalent (RCE) of a scalar random variable* $\mathbf{Z}$ *is defined as*

$$S_U(\mathbf{Z}) = \sup_z\{z + EU(\mathbf{Z} - z)\}.$$

where U *is a concave function.*

Applying Definition 2 to utility function (25) gives the RCE associated with this utility:

$$S_\beta(\mathbf{X}) = E(\mathbf{X}) - \frac{\beta}{2}Var(\mathbf{X}) \tag{26}$$

where β is a risk parameter. An agent maximizing the criterion in (26) is risk averse if $\beta > 0$.

Definition 3 *The recourse certainty equivalent (RCE) of the random sequence* $\mathbf{X} = (X_1, .., X_T)$ *is defined as [9]*

$$S_{\beta_1,..,\beta_T}(\mathbf{X}) = \sum_{t=1}^{T} b^{t-1}S_{\beta_t}(\mathbf{X}_t) = \sum_{t=1}^{T} b^{t-1}\{E(\mathbf{X}_t) - \frac{\beta_t}{2}Var(\mathbf{X}_t)\} \tag{27}$$

where the β_t *parameters allow to model different risk attitudes in different stages. The "utility" obtained at time t,* $S_{\beta t}$ *is defined as the difference:*

$$E(\mathbf{X}_t) - \frac{\beta_t}{2}Var(\mathbf{X}_t). \tag{28}$$

The definition of the period-t RCE in equation (28) corresponds to RCE in equation (26). An alternative motivation for the definition of period t objective in equation (28) is given in equation (24). Section 5 exemplifies the impact of risk on the optimal timing of investment.

5 Examples

Section 5.1 discusses examples of the optimal timing of the investment decision, based on expected value maximization. Section 5.2 extends these examples to accounting for risk. Sections 5.1-5.2 are based on the model introduced in 3.3. A comparison of the results in 5.1 and 5.2 reveals that risk may affect the optimal timing of the investment. In sections 5.1-5.2 the transition probabilities are independent of investment policies. In general, investments and investment subsidies have an impact on the income stream [6]. Section 5.3, based on section 3.2, discusses an example of a Markov decision process where the transition probabilities depend on the investment policy. Period-specific financial constraints explain the postponement of a part of the investments. Section 5.4 discusses an example of a partially observable MDP.

5.1 Expected Value Optimization

Consider a firm deciding its level of investment for $t = 1, .., 45$. For simplicity, the size of investment at all times is chosen from the discretized set $\{0, 1A, 2A, ..., 44A\}$ where the parameter $A > 0$ defines the value of the investment. Assume for simplicity there is no period-specific constraints on investment. Consider a discrete time model where the expected return at time t is defined as above as

$$E(r_t) = \sum_i r_{it} P_{it}. \tag{29}$$

In the absence of period-specific budget constraints $I_t \in \{0, I\}$, $\forall t$ and for a risk neutral decision maker with linear utility U the objective can be stated as:

$$\max\ U(\sum_{N=1}^{T} E(V(N))) = \max\ \sum_{N=1}^{T} E(V(N)) = \max_{N \in \{1,..,T\}} \{E(V(N))\}. \tag{30}$$

Thus, r_t observed at time t can be generated using the average return (29).

Assume first that the variation in the rate of return only depends on the future income subsidy, i.e. the other variations in income are assumed to be negligible. Simulations of the above Markov model were done assuming the transition probabilities between different states of return on investment

Table 3: Transition Probabilities before End of 3-Year Period

	0.3	0.2	0.16	0.11	0.08
0.3	0.01	0.3	0.4	0.28	0.01
0.2	0.01	0.8	0.1	0.09	0
0.16	0.01	0.05	0.7	0.15	0.09
0.22	0.01	0.01	0.08	0.8	0.1
0.08	0	0	0.05	0.15	0.8

(cf. Table 1) depend on the timing on the investment; the transition matrix before the end of a time period of three years is given in Table 3. The transition matrix after 3 years is given in Table 4. The underlying reasons for the change in transition probabilities include a potential change in income subsidies and a potential change in investment subsidies (affecting random income). Figure 1 illustrates the outcome in the case with $v = 0.09$ and $A = 5000$. Postponing investment gives the highest NPV for all levels of investment. The optimal solution is then to invest the maximum available amount at $t = 1$.

Note that the transition matrix before and after the policy change is defined assuming transitions take place within a given period of time (originally three years in Table 1). This affects the interpretation of v.

Time-Varying Income

Random income subsidies (adjusted by investment subsidies) was assumed to be the only source of variation in income in the above examples. In general, the income stream varies due to e.g. changes in production costs and demand.

Assuming a constant growth rate α in income, for a given return rate r, let $(1+r)^t = e^{\alpha t}$ denote income at t due to growth in value and consider an additive model:

$$E(r_t) = \sum_i P_{it} r_{it} + e^{\alpha t}.$$

The following summarizes the key results from simulating the model with

Table 4: Transition Probabilities after 3-Year Period

	0.3	0.2	0.16	0.11	0.08
0.3	0.0	0.2	0.4	0.3	0.1
0.2	0.0	0.7	0.15	0.15	0
0.16	0.0	0.1	0.6	0.2	0.1
0.11	0.0	0.01	0.19	0.7	0.1
0.08	0	0	0.0	0.05	0.95

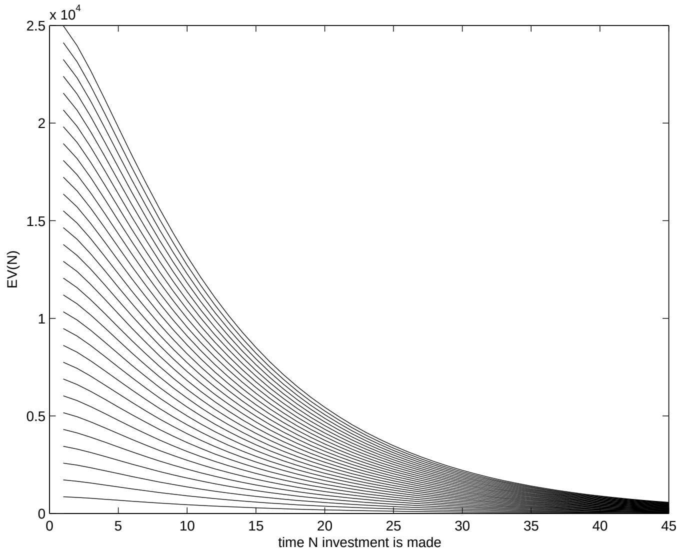

Figure 1: Expected net income at different levels of investment with $v = 0.09$ and $A = 5000$

different values for v and a constant α:

- If $\alpha > v$ then is pays off to postpone investment
- If $\alpha \leq v$ then it is optimal to invest at $t = 1$.

If α exceeds v, the $E(V(N))$-curves in Figure 1 become upward-sloping: it becomes optimal to postpone investment. It might be unrealistic to assume $\alpha > v$; nevertheless, there can be an incentive to postpone investment when taking risk into account.

5.2 Examples with Risk

Consider the optimal timing of investment assuming a binary investment decision: according to equation (30), the optimal time is determined as the time point when the expected net income or utility is highest. Figure 2 depicts a modification of the example in Figure 1. In particular, the utility in Figure 2 is defined according to Definition 3 in section 4, with $\beta/2 = (10)^{-4}$ (to ensure the utility is non-negative). The following can be observed from Figure 2, with a 45-years time horizon:

- Time-varying risk can affect the optimal timing of the investment; there can be a motivation for postponing the investment due to the time-varying variance.
- Time-varying risk only affects the optimal timing of relatively valuable investments. For investments greater than 110000 in value, the risk is high enough to make the net utility negative. For relatively low levels of investment, risk does not affect the optimal timing at $t = 1$.

5.3 Markov Decision Process

Recall the definition of a MDP in section 3: the transition matrix depends on the investment policy. (For example, [12] defines the transition probabilities for land price and debt as function of the investment policy.) To exemplify the optimal timing of investment with a MDP, consider the following example. Consider the model in 3.2, first without period-specific constraints as formalized in (15). This can be modelled via an MDP as follows. Assume

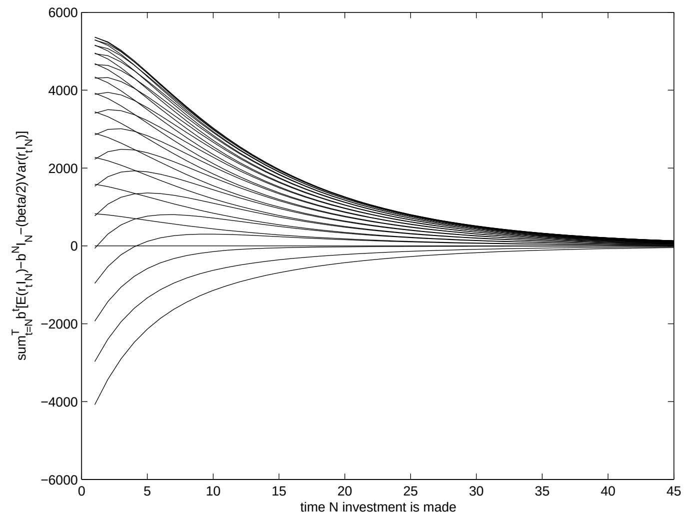

Figure 2: Expected net utility at different levels of investment with A=5000, $\beta/2 = 0.0001$, $v = 0.09$

that the transition matrix whenever investment has not been made is given in Table 5, appended by an additional state r_a corresponding to a state where the budget has been used up. After investment has been made a new transition probability matrix applies: one where each state leads to state r_a with probability one.

The immediate return when investing I_t at time t is given by

$$g_t = \frac{(r_t + e_t)I_t}{1-b} - I_t,$$

where e_t is a small random normal perturbation. In the absence of period-specific financial constraints an aggregate budget constraint I_a applies, i.e.:

$$\sum_t I_t \leq I_a. \tag{31}$$

With an aggregate budget constraint, the firm invests at most once. The probability of investment thus corresponds to the probability of investing I_a. In the following example let $I_a = 10000$.

Table 5: Transition Probabilities before End of 3-Year Period

	0.3	0.16	0.11	0.05	0
0.3	0.01	0.3	0.4	0.28	0.01
0.16	0.01	0.8	0.1	0.09	0
0.11	0.01	0.05	0.7	0.15	0.09
0.05	0.01	0.01	0.08	0.8	0.1
0	0	0	0.05	0.15	0.8

In this example, the only motivation for postponing investment is a starting state r_0 leaving scope for a slight improvement in r_t in future. This is true for both expected value maximization and expected utility maximization with period-specific utility given in equation (25). Figure 3 depicts the probability of investment with $v = 0.09$ and $\beta = 3(10)^{-5}$, showing only a marginal difference between a linear utility model and a nonlinear utility model for $t = 1, .., 30$ (after $t = 30$, the probability of investment is slightly higher with the linear utility). The similarity between the linear and nonlinear utility models was observed for different values of r_0 and b.

Let $Pr_t(I)$ denote the probability of investment at time t when the state g_t is observed. The expected value of perfect information (EVPI) can here be defined for each time period t as the weighted difference (cf. Definition 1):

$$EVPI(t) = E(\max\ b^t g_t)/Pr_t(I) - \max\{\max_t b^t E(g_t), 0\},$$

where the investment probability $Pr_t(I)$ is used as weight for $E(\max\ b^t g_t)$ (for the second term the corresponding probability is by definition equal to one). Figure 4 depicts the ratio between EVPI(t) and value of investment I_a. Assuming $v = 0.09$, $E(g_t)$ remains nonpositive for all t. The probability of investment at time t when maximizing the expected value of income stream is zero. The ratio depicted in figure 4 thus corresponds to the ratio

$$\frac{E(\max b^t g_t)}{Pr_t(I) I_a}. \tag{32}$$

The ratio (32) varies between 1.1 and 1.5 for $t = 1, .., 45$ (after $t = 45$ the probability of investment is less than 1 percent, explaining the peak after

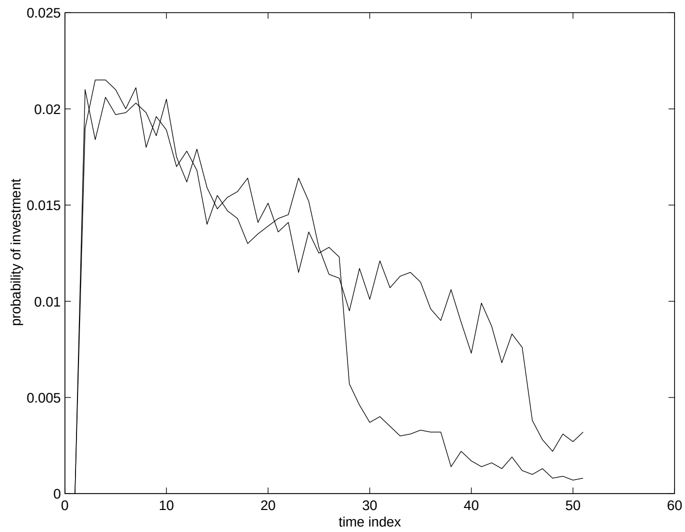

Figure 3: Probability of investment with aggregate budget constraint $I_a = 10000$, $v = 0.09$, $r_0 = 0.05$: linear and nonlinear utility model

$t = 45$). The value of perfect information EVPI(t) is high in this example: between 110 and 150 percent of total expenditure on investment. The same is true for $b = 0.95$ when $t = 1, ..., 7$; for $t = 8, .., 50$ the ratio $EVPI(t)/I_a$ gradually falls to 0. The mean value of the ratio over time in this latter case is 0.47.

Financial Constraints

Like in [12], assume now that the decision-maker decides at each time t on investment with period-specific constraints:

$$I_t \in \{0, I\},\ t = 1, .., T, \tag{33}$$

i.e. the total budget for investment with $T = 50$ periods is $50I$. Letting $I = 200$, the total maximum budget is 10000. Assume the state of the system r_t is a Markov process, with transition probabilities depending on the investment decision. Whether or not investment is not made, the transition probabilities are given in Table 5. The immediate return when investing I_t

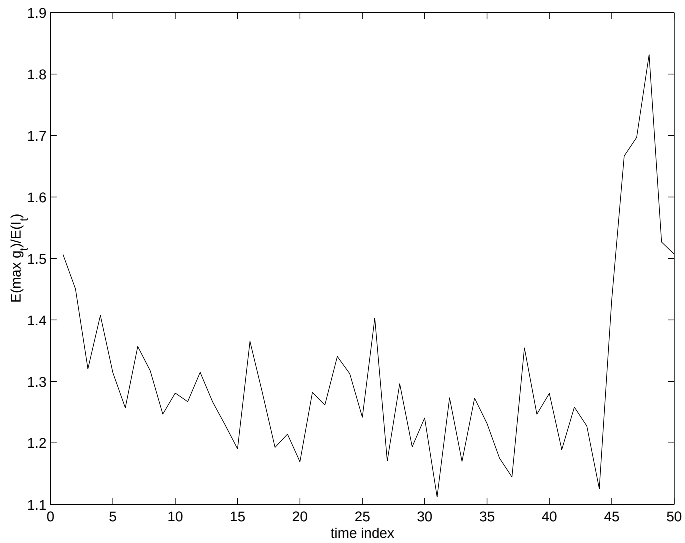

Figure 4: Ratio between EVPI and expected delay-discounted expenditure on investment, $v = 0.09$, $r_0 = 0.05$

at time t is given by

$$g_t = \frac{(r_t + e_t)I_t}{1 - b} - I_t,$$

where e_t is a small random normal perturbation. The total return to be maximized subject to (33) is

$$E(\sum_t b^t g_t).$$

The dynamic optimization problem can be solved by determining the time periods and states for which investment takes place. Set the continuous time interest rate v equal to 0.09. The above problem is solved numerically with backward recursion 10000 times, using Matlab [3]. The initial state is $r_0 = 0.11$. The mean optimal investment policy over time implies the probability that it is optimal to invest at $t = 1, .., T$, depicted in Figure 5. The outcome given in Figure 5 remains practically the same, when incorporating a risk-attitude via nonlinear utility functions (eq. (25)). This is similar to the above case without period-specific constraints.

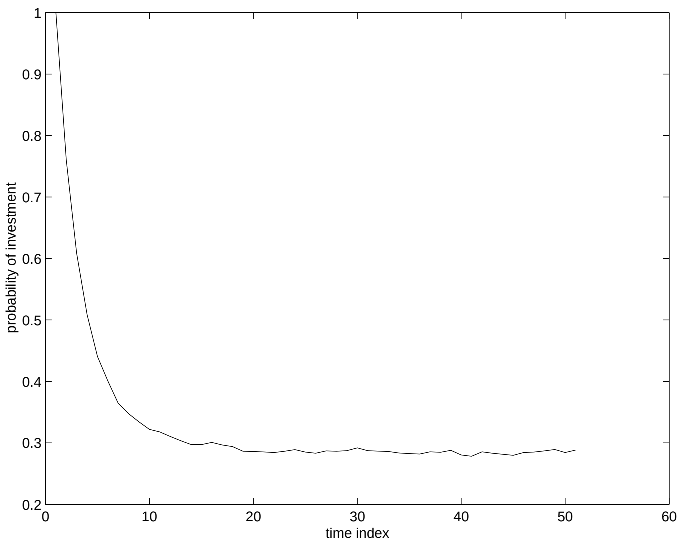

Figure 5: Probability of Investment, $v = 0.09$

The expected value of perfect information EVPI (Definition 1) is depicted in Figure 6. During the first three periods, the difference in EVPI(t)=$Ev(r_t) - v(E(r_t))$ is negligible: the optimal investment decision can be made based on expected state. At time $t = 4$, EVPI(4) is more than 15 percent of the delay-discounted value of total optimal investment $\sum_t b^t I_t$. After this, the cost percentage gradually decreases to zero over time.

5.4 Partially Observable Markov Process

In a discrete time version of the investment model in [2], the value r_{t+1} at time t can be assumed to depend on a weighted sum of a mean value μ and the value observed previous time period [5, 12, 2]:

$$r_{t+1} = \theta\mu + (1 - \theta)r_t + e_t \tag{34}$$

where e_t is a sequence of independent normally distributed random variables with zero mean. In many applications, the state of the system at time t is not fully observable. In this case the MDP model is called a Partially Observable MDP (POMDP) [10]. Assume that given the observed r_t, the decision maker

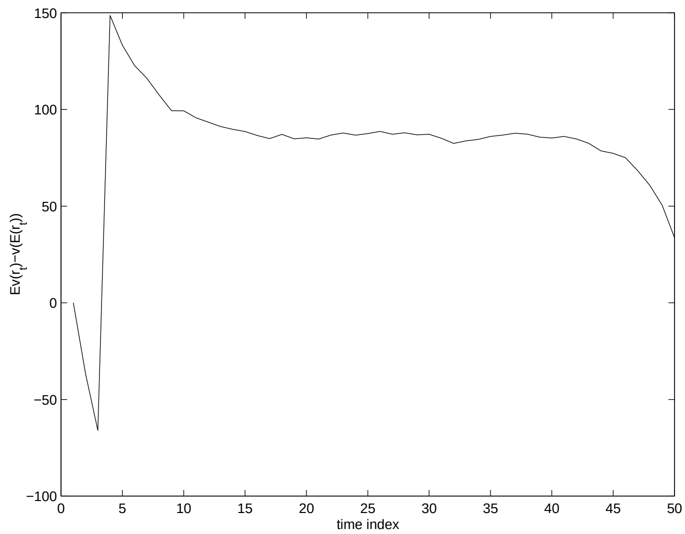

Figure 6: Value of perfect information, $v = 0.09$

forecasts the value r_{t+1} according to equation (34), assuming $\theta = 0$; thus, the process $\{r_t\}$ is a Markov process, perturbed by a small random variation. Consider a binary investment decision: $I_t \in \{0, I\}$ subject to an aggregate constraint:

$$\sum_t I_t \leq I.$$

If the decision-maker invests at time t, he obtains the value

$$W(t) = \frac{b^t(r_t + e_t)I}{1-b} - b^t I.$$

Consider the Markov process defined by transition probabilities in Table 5, perturbed by a small random normal disturbance. The utility value of investment is computed for each possible timing of investment $t = 1, .., 50$, taking the expectation and variance of the value of investment over 10000 runs, starting from state $r_0 = 0.16$. The solution to a risk-adjusted problem differs only marginally from the optimal solution of a risk-neutral decision maker. This is different from the results in section 5.1 where the variance is defined over sequences in time instead of over sequences at a given time. Figure 7 shows that for a risk-neutral decision-maker, it is optimal to invest at $t = 1$.

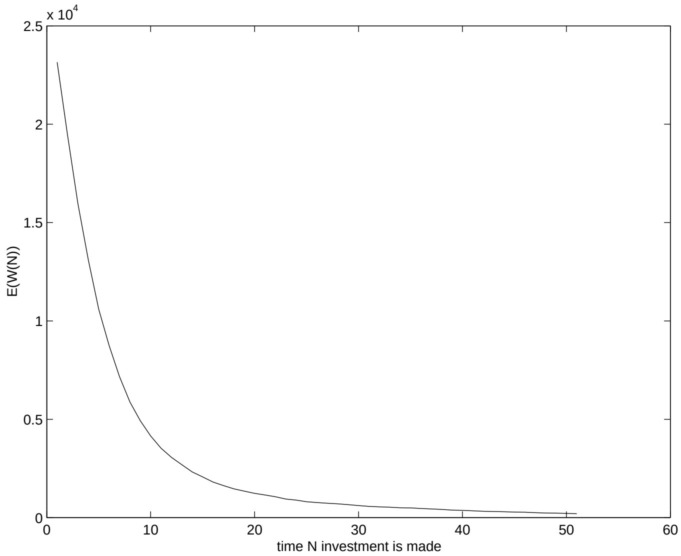

Figure 7: Expected income at different levels of investment with I=10000, $v = 0.04$

6 Conclusion

This paper has introduced a stochastic programming approach to optimal investment under uncertainty. Based on this approach, the cost of uncertainty regarding agricultural subsidies can be quantified. Numerical examples suggest that the cost of uncertainty can be significant. This implies that the efficiency of investment subsidy programs is deteriorated by the uncertainty regarding future income subsidies.

A positive option value is based on assuming an increasing expected value of the investment over time, assuming the decision-maker is risk-neutral. Explicitly accounting for risk helps to restore a positive option value, even if the random income is stationary or non-increasing in time. Numerical examples show that the optimal timing of the investment can be sensitive to the modelling of risk. A cost associated with risk (variance) is a source of an option value of postponing investment (along with period-specific financial constraints).

Topics for future work include the study of policy trade-offs (how much compensation is needed to overcome uncertainty costs) and the extension of the model to account for a random timing of a policy change.

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