Debreu's Coefficient of Resource Utilization, the Solow Residual, and TFP: The Connection by Leontief Preferences

14 January 2003

Thijs ten Raa[1]

Tilburg University
Box 90153
5000 LE Tilburg
the Netherlands
tenRaa@UvT.nl
Tel +31-20-6252240
Fax +31-20-4206502

Abstract. Debreu's coefficient of resource allocation is freed from individual data requirements. The procedure is shown to be equivalent to the imposition of Leontief preferences. The rate of growth of the modified Debreu coefficient and the Solow residual are shown to add up to TFP growth. This decomposition is the neoclassical counterpart to the frontier analytic decomposition of productivity growth into technical change and efficiency change. The terms can now be broken down by sector as well as by factor input.

JEL-codes: D24, O47

---

[1] This paper is an offspring of the workshop *Alternative measures of Total Factor Productivity growth* held on 13 September 2002 at the Department of Econometrics and Operations Research of Tilburg University and co-sponsored by the Netherlands Organization for Scientific Research (grant 460-02-031-A). Feedback from Victoria Shestalova is gratefully acknowledged.

1. Introduction

Total factor productivity (TFP) may grow by more efficient utilization of resources or by technical change. Debreu (1951) measured the utilization of resources and Solow (1957) measured technical change, but their models are remote. Solow's model is macro-economic and assumes perfect competition, while Debreu's model is micro-economic and assumes no technical change. In this paper I show how the measures of Debreu and Solow can be commingled into TFP. I take Debreu's model as point of departure, because it is quite general and, therefore, accommodating. The drawback of Debreu's coefficient of resource utilization, however, is that it hinges on individual preferences data. I will free his coefficient from this prohibitive data requirement, by making Debreu's concept of a 'better' commodity set independent of the specifics of individual preferences. The procedure will be shown to be equivalent to the adoption of Leontief preferences. The consequent 'tight' coefficient of resource utilization yields a more conservative estimate of inefficiency than Debreu's coefficient resource of utilization. As a bonus, the procedure makes the measure of inefficiency a function of total consumption only, not the individual breakdown. This paves the way for macro-economic applications and Solow residual analysis. The Solow residual is generalized to Debreu's setting.

Neoclassical economics encounters some refreshing competition from frontier analysis. See Färe, Grosskopf, Norris and Zhang (1994), the references given there, and Färe and Grosskopf (1996). This literature pays little attention to the marginal productivities of inputs and, therefore, does not ascribe TFP to factors, but it provides a useful decomposition of productivity growth into technical change and efficiency change. I take this idea into the neoclassical realm. The connection is at a rather abstract level, for the mechanisms behind efficiency change are different in frontier analysis and neoclassical economics. Frontier analysis captures technological catch-up with the leader and the choice of inputs in terms of costs. Neoclassical analysis captures potential reallocations of resources between sectors. This type of efficiency change is harder to detect. Frontier analytic inefficiency is exposed by the gap with the best practice, a conceptually straightforward concept. Neoclassical inefficiency, however, not only comprises gaps with production possibility frontiers, but also hidden misallocations. A contribution of this paper is that it shows how the tools of frontier analysis, particularly the input- and output-distance functions, can be applied to the measurement of allocative efficiency.

The pieces of the puzzle fit pleasingly well. More precisely, in this paper I show that total factor productivity growth is the sum of technical change and efficiency change, where the former is the (generalized) Solow residual and the latter is the rate of growth of the (tight) coefficient of resource utilization.

2. Debreu's coefficient of resource utilization

Debreu (1951) measures the inefficiency of the allocation of resources in an economy by calculating how much less resources could attain the same level of satisfaction to the consumers. I will review his so-called coefficient of resource utilization.

The economy comprises m consumers with preference relationships $\succsim_i$ and observed consumption vectors $\mathbf{x}^0_i \in \mathbb{R}^l$ (i = 1, ..., m), where l is the number of commodities. $Y \subset \mathbb{R}^l$ is the set of possible input vectors (*net* quantities of commodities consumed by the whole production sector during the period considered), including the observed one, $\mathbf{y}^0$. A combination of consumption vectors and an input vector is *feasible* if the total sum—the economy-wide *net* consumption—does not exceed the vector of *utilizable physical resources*, $\mathbf{z}^0$.[2] Vector $\mathbf{z}^0$ is assumed to be at least equal to the sum of the observed consumption and input vectors, ensuring the feasibility of the latter.

The set of net consumption vectors that are at least as good as the observed ones is

$$\mathcal{B} = \{\textstyle\sum \mathbf{x}_i \mid \mathbf{x}_i \succsim_i \mathbf{x}^0_i,\ i=1,\dots,m\} + Y \qquad (1)$$

The symbol $\mathcal{B}$ stands for 'better' set. The minimal resources required to attain the same levels of satisfaction that come with $\mathbf{x}^0_i$ belong to $\mathcal{B}^{\min}$, the South-western edge

[2] For example, if the last commodity, l, represents labor, and this is the only nonproduced commodity, then $\mathbf{z}^0 = Ne_l$, where N is the labor force and e_l the l-th unit vector.

or subset of elements that are minimal with respect to $\geqq$.[3] Assume that preferences are convex and continuous, and that production possibilities form a convex and closed set, then the separating hyperplane theorem yields a supporting price vector $\mathbf{p}(\mathbf{z}) > \mathbf{0}$ such that $\mathbf{z}' \in \mathcal{B}$ implies $\mathbf{p}(\mathbf{z})\cdot\mathbf{z}' \geqq \mathbf{p}(\mathbf{z})\cdot\mathbf{z}$.[4] At this junction Debreu (1951) defines the *coefficient of resource allocation* by

$$\rho = \text{Max } \mathbf{p}(\mathbf{z})\cdot\mathbf{z}/\mathbf{p}(\mathbf{z})\cdot\mathbf{z}^0 \text{ subject to } \mathbf{z} \in \mathcal{B}^{\min} \qquad (2)$$

Coefficient ρ measures the distance from the set of minimally required physical resources, $\mathbf{z} \in \mathcal{B}^{\min}$, to the utilizable physical resources, $\mathbf{z}^0$, in the metric of the supporting prices (which indicate welfare indeed). Debreu (1951, p. 284) shows that the distance or the Max in (2) is attained by[5]

$$\mathbf{z} = \rho\mathbf{z}^0 \in \mathcal{B}^{\min} \qquad (3)$$

In other words, the coefficient of resource allocation is the smallest fraction of the actually available resources that would permit the achievement of the levels of satisfaction that come with $\mathbf{x}^0_i$. Coefficient ρ is a number between zero and one, the

---

[3] By convention, this vector inequality holds if it holds for all components.

[4] $\mathbf{p} > \mathbf{0}$ means that all components are positive. The prices are positive because $\mathbf{z} \in \mathcal{B}^{\min}$ and it is the only point in common to $\mathcal{B}$ and $\{\mathbf{z}' | \mathbf{z}' \leqq \mathbf{z}\}$; hence $\mathbf{p}$ may be chosen such that $\mathbf{p}\cdot\mathbf{z}' < \mathbf{p}\cdot\mathbf{z}$ for $\mathbf{z}' \leqq \mathbf{z}$ (except $\mathbf{z}' = \mathbf{z}$).

[5] There are two, related caveats in Debreu's (1951) analysis: $\mathbf{z} = \rho\mathbf{z}^0 \in \mathcal{B}^{\min}$ need not exist and ρ may not be unique if the separating price vector is not unique. Consider an economy with two commodities, one consumer, and no production (or $Y = \{0\}$). $\mathbf{x} \succsim \mathbf{x}'$ if $\min(x_1,x_2) \geqq \min(x'_1,x'_2)$. $\mathbf{x}^0 = (1\ 1)$ and $\mathbf{z}^0 = (1\ 2)$. Then $\mathcal{B}^{\min} = \{\mathbf{x} | \mathbf{x} \succsim \mathbf{x}^0\}^{\min} = \{\mathbf{x} | \mathbf{x} \geqq (1\ 1)\}^{\min} = \{(1\ 1)\}$ contains no $\rho\mathbf{z}^0 = \rho(1\ 2)$. What is the coefficient of resource utilization? In this case, any $\mathbf{p} > \mathbf{0}$ separates $\mathcal{B} = \{\mathbf{x} | \mathbf{x} \geqq (1\ 1)\}$ and $\{\mathbf{z}' | \mathbf{z}' \leqq (1\ 1)\}$; hence (2) yields $\rho = \mathbf{p}\cdot(1\ 1)/\mathbf{p}\cdot(1\ 2) = (p_1 + p_2)/(p_1 + 2p_2)$, a number between 0.5 and 1. To resolve the multiplicity, we may address the efficiency problem in primal space. The preference relationship is represented by utility $\min(x_1,x_2)$. Subject to feasibility constraint $\mathbf{x} \leqq \mathbf{z}^0 = (1\ 2)$, the maximum utility is 1. This is attained by $\mathbf{x}^0 = (1\ 1)$. Hence the allocation is optimal. Following Debreu's (1951, p. 275) introduction, $\rho = 1$. This implies that $\mathbf{p} = (1\ 0)$. Indeed, this is the supporting price system of the second welfare theorem. However, it is not positive.

If the minimal $\rho\mathbf{z}^0$ belongs to $\mathcal{B}^{\min}$, then the prices in (2) are positive and the coefficient ρ generated by (4) solves (2), following Debreu (1951, p. 284). If the minimal $\rho\mathbf{z}^0$ does not belong to $\mathcal{B}^{\min}$, the prices in (2) are only nonnegative, but the coefficient ρ generated by (4) still solves (2).

latter indicating full efficiency. In modern terminology, this result means that ρ is the *input-distance function*, determined by the program

$$\text{Min } \rho \text{ subject to } \sum \mathbf{x}_i + \mathbf{y} \leqq \rho \mathbf{z}^0,\ \mathbf{x}_i \succsim_i \mathbf{x}^0_i,\ \mathbf{y} \in Y \tag{4}$$

There is also an output-distance function, but that one is opaque. The measurement of satisfaction is in terms of utility, an ordinal concept that generally admits no aggregation over consumers.

3. Absent individual data: the tight coefficient of resource utilization

Following Debreu (1951) a simple symbol $\mathcal{B}$ has been used to denote the 'better set.' Definition (1) reveals, however, that the set depends on the observed consumption vectors and on preferences. The informational requirements involved are prohibitive. To overcome this problem, I will define the *tight coefficient of resource allocation*, ρ^*. Basically, I will require that the notion of 'better' set will be independent of the specifics of preferences. For this purpose, all I assume is that preferences are weakly monotonic in the sense that they belong to

$$\mathcal{M} = \{\succsim \mid (\mathbf{x}' \geqq \mathbf{x}) \text{ implies } (\mathbf{x}' \succsim \mathbf{x})\} \tag{5}$$

I now define the tight better set as the intersection of all better sets over $\mathcal{M}$:

$$\mathcal{B}^* = \cap\{\sum \mathbf{x}_i \mid \mathbf{x}_i \succsim \mathbf{x}^0_i,\ i=1,\ldots,m\} + Y \tag{6}$$

The replacement of the better set, $\mathcal{B}$, by the tight better set, $\mathcal{B}^*$, implies that definition (2) produces ρ^* instead of ρ. A comparison between these two coefficients is obtained by rewriting program (4):

$$\text{Min } \rho^* \text{ subject to } \sum \mathbf{x}_i + \mathbf{y} \leqq \rho \mathbf{z}^0,\ \mathbf{x}_i \succsim \mathbf{x}^0_i \text{ for all } \succsim\ \in \mathcal{M},\ \mathbf{y} \in Y \tag{7}$$

The constraint set of (7) is contained in the one of (4); hence the solution to program (4) must be sharper:

$$\rho \leqq \rho^* \tag{8}$$

In other words, use of the tight better set will overestimate efficiency or underestimate inefficiency. Debreu's (1951) measure of inefficiency reflects scope for reallocation of resources between consumers with different tastes and, therefore, is quite high; the flipside of this observation is that his coefficient of resource allocation is relatively low.

4. The tight coefficient of resource utilization and Leontief preferences

I can be a bit more specific about the tight coefficient of resource allocation. I will show that it is generated by Leontief preferences. Leontief preferences $\precsim(\mathbf{a})$ with nonnegative bliss point $\mathbf{a} \in \mathbb{R}^l$ are defined for nonnegative consumption vectors by $\mathbf{x}' \succsim(\mathbf{a})\ \mathbf{x}$ if $\min x_k'/a_k \geqq \min x_k/a_k$ where the minimum is taken over commodities $k = 1,...,l$. The minima exist if $\mathbf{a}$ is nonzero, what I assume.[6]

Lemma. $\mathcal{B}^* = \{\sum \mathbf{x}_i \mid \mathbf{x}_i \succsim(\mathbf{x}^0{}_i)\ \mathbf{x}^0{}_i,\ i=1,...,m\} + Y = \{\mathbf{x} \mid \mathbf{x} \geqq \sum \mathbf{x}^0{}_i\} + Y.$

Proof. I show that the first term of $\mathcal{B}^*$ in (6) is contained in the first term of the second set, that the latter is contained in the first of the third set, and that last one in the first term of $\mathcal{B}^*$. Using the fact $\succsim(\mathbf{x}^0{}_i) \in \mathcal{M}$, I have $\cap\{\sum \mathbf{x}_i \mid \mathbf{x}_i \succsim_i\ \mathbf{x}^0{}_i,\ i=1,...,m\} \subset \{\sum \mathbf{x}_i \mid \mathbf{x}_i \succsim(\mathbf{x}^0{}_i)\ \mathbf{x}^0{}_i,\ i=1,...,m\} = \{\sum \mathbf{x}_i \mid \mathbf{x}_i \geqq \mathbf{x}^0{}_i\} = \{\mathbf{x} \mid \mathbf{x} \geqq \sum \mathbf{x}^0{}_i\} \subset \cap\{\sum \mathbf{x}_i \mid \mathbf{x}_i \succsim_i\ \mathbf{x}^0{}_i,\ i=1,...,m\}$, where the last inclusion is shown as follows. For $\mathbf{x} \geqq \sum \mathbf{x}^0{}_i$, allocate the

[6] Situations like labor supply are covered by letting the commodity be leisure time. Division by zero is assumed to yield infinity.

surplus in any nonnegative way, for example by putting $\mathbf{x}_1 = \mathbf{x}^0_1 + \mathbf{x} - \sum\mathbf{x}^0_i$, $\mathbf{x}_2 = \mathbf{x}^0_2$, ..., $\mathbf{x}_m = \mathbf{x}^0_m$, then $\mathbf{x}_i \succsim_i \mathbf{x}^0_i$ for all $\succsim_i \in \mathcal{M}$. Q.E.D.

The first equality in the Lemma implies that if the consumers have Leontief preferences, then the coefficient of resource allocation is tight. The second equality in the Lemma frees the better set, $\mathcal{B}$, from preferences, $\succsim_i$, as well as from individual consumption baskets, $\mathbf{x}^0_i$. The tight better set, $\mathcal{B}^*$, depends only on the *total consumption vector*, $\sum\mathbf{x}^0_i$. This modification facilitates measurement of the coefficient of resource utilization. In fact, the tightening creates the option to determine the degree of efficiency in terms of outputs, resurrecting the output-distance function.

Corollary. Assume that the total consumption vector $\sum\mathbf{x}^0_i$ is nonnegative and nonzero. Assume that the production set Y features the impossibility to produce something from nothing and constant returns to scale. Then $c = 1/\rho^*$ transforms the input-distance function program (7) into the *output-distance function* program

$$\text{Max } c \text{ subject to } c\sum\mathbf{x}^0_i + \mathbf{y} \leqq \mathbf{z}^0,\ \mathbf{y} \in Y$$

Proof. By the Lemma, program (7) can be rewritten as

$$\text{Min } \rho^* \text{ subject to } \sum\mathbf{x}_i + \mathbf{y} \leqq \rho\mathbf{z}^0,\ \mathbf{x}_i \succsim(\mathbf{x}^0_i)\ \mathbf{x}^0_i,\ i=1,...,m,\ \mathbf{y} \in Y$$

or

$$\text{Min } \rho^* \text{ subject to } \mathbf{x} + \mathbf{y} \leqq \rho\mathbf{z}^0,\ \mathbf{x} \geqq \sum\mathbf{x}^0_i,\ \mathbf{y} \in Y$$

This can be simplified further to

$$\text{Min } \rho^* \text{ subject to } \sum\mathbf{x}^0_i + \mathbf{y} \leqq \rho\mathbf{z}^0,\ \mathbf{y} \in Y$$

The solution is positive. (Otherwise input vector $\mathbf{y} \leqq -\sum\mathbf{x}^0_i \leqq 0$, but not equal to zero, would produce something from nothing.) The transformation is completed by multiplication by $c = 1/\rho^*$ and a change of variable ($c\mathbf{y}$ to $\mathbf{y}$), using constant returns to scale. Q.E.D.

The output-distance function program informs us by which factor the total consumption vector can be expanded, given the resources.

5. Application

The Corollary shows that under constant returns to scale the inverse of the tight coefficient of resource utilization is the expansion factor of the economy, *c*. ten Raa (1995) calculates *c* for the Canadian economy, defines $1 - 1/c$ as inefficiency, and decomposes the latter into productive inefficiency, allocative inefficiency, and trade inefficiency. It follows that ten Raa's (1995) measure of inefficiency is $1 - \rho^*$. In view of inequality (8), this measure of inefficiency underestimates Debreu's (1951) degree of inefficiency, $1 - \rho$. Debreu (1951) finds more scope for efficiency gains as marginal rates of substitution may be equalized across consumers. The tight coefficient of resource utilization does not take into account this source of inefficiency.

ten Raa (1995) and ten Raa and Mohnen (2002) divide the commodities between produced goods and factor inputs, respectively. **U** is a table depicting the use of goods by sectors and **V** is a table depicting the outputs of the sectors in terms of goods. $\mathbf{U} - \mathbf{V}^\top$ is the net input table; its dimension is that of goods by sectors.[7] **L** is the factor input table; its dimension is factor inputs by sectors. An element of $\mathbf{y} \in Y$ has components $(\mathbf{U} - \mathbf{V}^\top)\mathbf{s}$ and $\mathbf{Ls}$, and Y is defined by letting the allocation vector $\mathbf{s} \geqq 0$. Similarly, $\sum \mathbf{x}^0_i$ has components **f** and **0**, where **f** is the vector of final goods consumption, while $\mathbf{z}^0$ has components **0** and **N**, where **N** is the vector of factor endowments, and **z** has components **0** and **Ls**. The output-distance function program of the Corollary becomes

$$\text{Max}_{c,\mathbf{s}}\ c \text{ subject to } c\mathbf{f} + (\mathbf{U} - \mathbf{V}^\top)\mathbf{s} \leqq \mathbf{0},\ \mathbf{Ls} \leqq \mathbf{N},\ \mathbf{s} \geqq \mathbf{0} \qquad (9)$$

[7] Superscript $^\top$ denotes transposition.

The solution to this program yields the potential standard of living, relative to the observed one.[8] The shadow prices of the second constraint yield the factor productivities.

6. Relationship with the Solow Residual

This section is the centerpiece of the paper. The tight coefficient of resources and a generalized Solow residual are tied up into total factor productivity growth. What is productivity?

An economy transforms physical resources into final consumption by means of production. The ratio of consumption to the resources is called the *productivity* of the economy. Productivity may grow because the production possibility set increases or because resources are better utilized. Productivity growth equals the sum of technical change and efficiency change. Technical change is the shift of the production possibility frontier and efficiency change the raise of the coefficient of resource utilization. That the two add to productivity growth is a well-known fact in frontier analysis, such as Data Envelopment Analysis, but that literature is mechanical in that the relative contributions of factor inputs are not valued. Neoclassical growth accounting has this feature, and also the capacity to decompose technical change by sector, but ignores inefficiency and its fluctuation. Recently ten Raa and Mohnen (2002) reconciled the two approaches in a single input-output framework, featuring international trade. I will now uncover the relationship at the level of generality of the Debreu model.

---

[8] Of course, any positive coefficient may be entered in the objective function and this is commendable, to scale the price level. As is, by the main theorem of linear programming, the factor input shadow prices fulfill $\mathbf{w}\cdot\mathbf{N} = c$. Since c is of the order one but $\mathbf{N}$ of the order millions, $\mathbf{w}$ will be tiny. A handy objective function is $\mathbf{e}.\mathbf{f}\ c$, where $\mathbf{e}$ is the unit vector with all components one. The dual constraints then show that $\mathbf{p}\cdot\mathbf{f} = \mathbf{e}.\mathbf{f}$ and $\mathbf{w}\cdot\mathbf{N} = \mathbf{p}\cdot c\mathbf{f}$. In other words, the product prices are normalized at unity and the factor input prices fulfill the potential national income identity. The proof is as follows. Multiplication of the dual constraint associated with variable $\mathbf{s}$, by $\mathbf{s}$, yields $\mathbf{p}(\mathbf{U} - \mathbf{V}^{\top})\mathbf{s} + \mathbf{wLs} = 0$. Replace the two terms using the two respective constraints of program (14): $-\mathbf{p}c\mathbf{f} + \mathbf{wN} = 0$, where (priced) inequalities are binding according to the phenomenon of complementary slackness. The product price normalization follows by the main theorem of linear programming or $\mathbf{w}\cdot\mathbf{N} = \mathbf{e}.\mathbf{f}\ c$.

The point of departure is the tight coefficient of resource allocation (ρ^*), determined by program (7) or, using the Lemma,[9]

$$\text{Min}\ \rho^* \text{ subject to } \sum \mathbf{x}^0_i + \mathbf{y} \leqq \rho \mathbf{z}^0,\ \mathbf{y} \in Y \tag{10}$$

. Assuming free disposal, input may be added to $\mathbf{y} \in Y$ until the constraint is binding:

$$\sum \mathbf{x}^0_i + \mathbf{y} = \rho^* \mathbf{z}^0 \tag{11}$$

This is the material balance.

Let $\mathbf{p}$ support the tight better set defined in (5), $\mathcal{B}^*$, in the sense introduced before (2).[10] According to the phenomenon of complementary slackness, non-linear program (10) yields

$$\mathbf{p} \cdot \sum \mathbf{x}^0_i = \rho^*\ \mathbf{p} \cdot \mathbf{z}^0 - \mathbf{p} \cdot \mathbf{y} \tag{12}$$

This is the identity between national product and national income; it holds even when there is no free disposal and, therefore, the material balance, (11), is not fulfilled. The national product is on the left hand side and on the right hand side is factor income plus profit. (Remember, $\mathbf{y}$ is net input, hence $-\mathbf{y}$ is net output.) All this is at the optimum allocation ($\sum \mathbf{x}^0_i$, $\mathbf{y}$, $\rho^* \mathbf{z}^0$) and supporting (or competitive) prices $\mathbf{p}$, not the actual allocation ($\sum \mathbf{x}^0_i$, $\mathbf{y}^0$, $\mathbf{z}^0$) and prices.

The economy transforms resources $\mathbf{z}^0$ into consumption $\sum \mathbf{x}^0_i$. The ratio of the latter to the former constitutes the level of total factor productivity. Since the objects are vectors, they must weighed by prices, for which $\mathbf{p}$ is employed. The *level* of total factor productivity is thus $\mathbf{p} \cdot \sum \mathbf{x}^0_i / \mathbf{p} \cdot \mathbf{z}^0$. If there are constant returns to scale, profit is zero, and, by equation (12):

$$\mathbf{p} \cdot \sum \mathbf{x}^0_i / \mathbf{p} \cdot \mathbf{z}^0 = \rho^* \tag{13}$$

[9] See the proof of the Corollary.

[10] Footnote 4 shows that the supporting prices are not necessarily positive.

This equation shows that the level of total factor productivity is equal to the tight coefficient of resource allocation.

Another interesting connection is the following. Let all variables vary with time and let d denote a time derivative. *Total factor productivity growth* is the rate of growth of the level of total factor productivity at *fixed* price weights:

$$TFP = \mathbf{p}\cdot d\sum \mathbf{x}^0{}_i/\mathbf{p}\cdot\sum \mathbf{x}^0{}_i - \mathbf{p}\cdot d\mathbf{z}^0/\mathbf{p}\cdot\mathbf{z}^0 \qquad (14)$$

The following proposition shows that TFP is the sum of the Solow residual, generalized to Debreu's framework, and the rate of growth of the tight coefficient of resource utilization. The *generalized Solow residual* is defined by

$$SR = -\mathbf{p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum \mathbf{x}^0{}_i \qquad (15)$$

This expression features the change in optimal net output, **-y**, and will be shown to be a generalized Solow residual indeed, evaluated at the frontier. The demonstration is in the next section, where the residual will be shown to measure the shift of the production possibility function.

Proposition. Under constant returns to scale, $TFP = SR + d\rho^*/\rho^*$.

Proof. Under constant returns to scale equation (13) holds. Substitution in equation (14) yields $TFP = [\mathbf{p}\cdot d\sum \mathbf{x}^0{}_i - \rho^*\mathbf{p}\cdot d\mathbf{z}^0]/\mathbf{p}\cdot\sum \mathbf{x}^0{}_i$. Substitution of the material balance, (11), and the product rule yield $TFP = [-\mathbf{p}\cdot d\mathbf{y} + \mathbf{p}\cdot\mathbf{z}^0 d\rho^*]/\mathbf{p}\cdot\sum \mathbf{x}^0{}_i$. Substitution of equations (13) and (15) yields the posted formula. Q.E.D.

The first TFP term, SR, reflects *technical change*. The second TFP term, $d\rho^*/\rho^*$, is the rate of growth of the tight coefficient of resource utilization and, therefore, represents *efficiency change*. The decomposition of productivity growth in technical and efficiency changes is inspired by frontier analysis. That literature, however, does not relate total factor productivity with the marginal productivities of the inputs. The neoclassical growth accounting literature does accomplish this, but at the price of assuming efficiency, ignoring the efficiency change term.

7. The Solow residual

Solow (1957) divides commodities between a single output and factor inputs. Denoting the latter by a vector **l**, the producible output is $F(\mathbf{l}, t) - s$, where $F(\cdot, t)$ is the production function at time t (presumed quasi-concave) and s is slack.[11] A net input vector $\mathbf{y} \in Y$ has components $-F(\mathbf{l}, t) + s$ and **l**, respectively. The production possibility set Y is obtained by letting $\mathbf{l} \geqq \mathbf{0}$ and $s \geqq 0$. The vector of available resources, $\mathbf{z}^0$, has components 0 and $\mathbf{l}^0$, respectively. Let ρ^* be the tight coefficient of resource utilization and **y** be the optimal net input vector, which solve efficiency program (10), then **y** has components $-F(\mathbf{l}, t)$ and $\mathbf{l} = \rho^* \mathbf{l}^0$. The first or product component of the material balance, (11), reads

$$(\sum \mathbf{x}^0_i)_1 - F(\mathbf{l}, t) = \rho^* 0 = 0 \tag{16}$$

The other or factor components read

$$0 + \mathbf{l} = \rho^* \mathbf{l}^0 \tag{17}$$

An intuitive interpretation of the tight coefficient of resource utilization, ρ^*, is in terms of actual output, $F(\mathbf{l}^0, t) - s^0$, where $F(\mathbf{l}^0, t)$ is potential output and s^0 is observed slack. Actual output could also be generated by optimal factor input **l** (with no slack). It follows that the actual/potential output ratio is $F(\mathbf{l}, t)/F(\mathbf{l}^0, t)$. By equation (17), this is ρ^* if the production has constant returns to scale. *The tight coefficient of resource utilization is the ratio of actual to potential output.*

As is well known, the solution **y** is supported by price vector $(1\ \mathbf{w}) = (1\ \partial_{\mathbf{l}}F(\mathbf{l}, t))$ where ∂ denotes partial derivatives (with respect to **l** in this case) or marginal

---

[11] Slack scalar s should not be confused with allocation vector **s** of section 5.

productivities.[12] I will now evaluate $SR = \mathbf{-p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i$ of definition (15) for this special setting. The numerator reduces to

$$\mathbf{-p}\cdot d\mathbf{y} = dF(\mathbf{l}, t) - \mathbf{w}d\mathbf{l} \tag{18}$$

and the denominator is, in view of the first terms of (16) and (17),

$$\mathbf{p}\cdot\sum\mathbf{x}^0_i = (\sum\mathbf{x}^0_i)_1 + 0 = F(\mathbf{l}, t) \tag{19}$$

using (16). Hence the quotient is

$$SR = \mathbf{-p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i = dF(\mathbf{l}, t)/F(\mathbf{l}, t) - \sum_k[w_k l_k/F(\mathbf{l}, t)]dl_k/l_k \tag{20}$$

The expression on the right hand side is, indeed, the residual between the output growth rate and the input growth rates, where value shares weight the latter. The shares add up if the production function has constant returns to scale, by Euler's theorem. The input prices are competitive marginal productivities, which are high in the sense that they leave no room for profit. The use of lower, observed prices, will bias upward expression (20).

The main point of Solow (1957) was that the residual measures technical change, a result that is easily verified. By equation (18), $\mathbf{-p}\cdot d\mathbf{y} = dF(\mathbf{l}, t) - \partial_{\mathbf{l}}F(\mathbf{l}, t)d\mathbf{l}$. But by total differentiation, $dF(\mathbf{l}, t) = \partial_{\mathbf{l}}F(\mathbf{l}, t)\, d\mathbf{l} + \partial_t F(\mathbf{l}, t)$. Hence the numerator of $SR = \mathbf{-p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i$, see definition (15), simplifies to $\partial_t F$ and we obtain, using equation (19),

$$SR = \mathbf{-p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i = \partial_t F(\mathbf{l}, t)/F(\mathbf{l}, t) \tag{21}$$

The Solow residual measures the relative shift of the production function indeed.

Residual expression (20) can be generalized to multi-products. Then the output growth term is an output-value share weighted expression. Intermediate products can

---

[12] If $F(\cdot, t)$ is not differentiable, a subgradient will do.

also be accommodated; this will be detailed in the next section. All are encompassed by definition (15): $SR = -\mathbf{p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i$, where $-\mathbf{y}$ is resource minimizing net output and $\sum\mathbf{x}^0_i$ is observed total consumption.

For constant returns to scale, the minimization of resources subject to total consumption—see program (10)—amounts to a maximization of consumption subject to available resources—program (9). As was shown there, this merely involves a change of variable from $\mathbf{y}$ to $c\mathbf{y}$ and a replacement of $\sum\mathbf{x}^0_i$ by $c\sum\mathbf{x}^0_i$. The expansion factors c in the numerator and in the denominator of the generalized Solow residual, $SR = -\mathbf{p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i$, cancel and its expression may therefore be reinterpreted in terms *maximal* consumption and sustaining optimal net output. The maximum consumption vector has the same proportions as the observed consumption vector. The prices in the generalized Solow residual are not affected at all, because of the constant returns to scale.

8. Productivity and efficiency decompositions

There are two further decompositions of total factor productivity growth than in technical change and efficiency change. The first decomposition is in factor productivity growth rates; it sounds dull, but is not achieved in frontier analysis. The second decomposition is by input-output sector.

The decomposition by *factor* is standard neoclassical analysis, at least for the Solow residual. Assume constant returns to scale, then $\mathbf{p}\cdot\mathbf{y} = 0$ and the generalized Solow residual becomes

$$SR = -\mathbf{p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i = d\mathbf{p}\cdot\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i \qquad (22)$$

Remember, $\mathbf{y}$ is the vector of net inputs. $\mathbf{p}$ is the vector of shadow prices or marginal productivities. Equation (22) imputes the technical change term of total factor productivity to the various inputs. It is very general. It reduces to the more familiar Jorgenson and Griliches (1967) form in the Solow world with a macro-economic production function, introduced in the previous section. There $\mathbf{y}$ has components $-F(\mathbf{l},$

t) and $\mathbf{l}$, and $\mathbf{p} = (1\ \mathbf{w}) = (1\ \partial_{\mathbf{l}}F(\mathbf{l}, t))$. Hence the numerator of the generalized Solow residual (22) reduces to $d\mathbf{p}\cdot\mathbf{y} = d\mathbf{w}\cdot\mathbf{l}$, while the denominator is $F(\mathbf{l}, t)$ by equation (19). It follows that the Solow residual becomes

$$SR = -\mathbf{p}\cdot d\mathbf{y}/\mathbf{p}\cdot\sum\mathbf{x}^0_i = \sum_k[w_k l_k/F(\mathbf{l}, t)]dw_k/w_k \qquad (23)$$

The expression on the right hand side is the growth rate of the factor productivity, with components weighted by their value shares. The input prices are competitive marginal productivities, which are high in the sense that they leave no room for profit. The use of lower, observed prices, will bias downward expression (23), unlike expression (20), which was biased upward in this case. The (primal) expression (20) and the (dual) expression (23) thus provide inconsistent estimates when no competitive prices are used.

The inclusion of efficiency change amounts to proportional increases of the factor productivity growth rates. By Debreu's equation (3), the minimally required physical resources, $\mathbf{z}$, are proportional to the utilizable physical resources, $\mathbf{z}^0$.[13] In the world Solow this proportionality is between minimal factor inputs $\mathbf{l}$ and observed factor inputs $\mathbf{l}^0$, see equation (16), or $\rho^* = l_k/l^0_k$, all k. Hence the efficiency change term of TFP reads, assuming constant returns to scale,

$$d\rho^*/\rho^* = \sum_k[w_k l_k/F(\mathbf{l}, t)]\ d\rho^*/\rho^* = \sum_k[w_k l_k/F(\mathbf{l}, t)](dl_k/l_k - dl^0_k/l^0_k) \qquad (24)$$

Substituting expressions (23) and (24) into the Proposition (section 6), all TFP-growth is now decomposed in terms of factor contributions:

$$TFP = \sum_k[w_k l_k/F(\mathbf{l}, t)](dw_k/w_k + dl_k/l_k - dl^0_k/l^0_k) \qquad (25)$$

The leading term measures factor productivity growth and the remainder the factor utilization growth. For each factor, the value share of the factor weights the sum of the two growth measures.

[13] See the disclaimer in footnote 4 though.

The generalized Solow residual is decomposed by *sector* by adding the structure of section 5. What follows is an activity variant of Hulten's (1978) analysis. In section 5, the net input vector **y** sustaining maximal consumption has components $(\mathbf{U} - \mathbf{V}^{\top})\mathbf{s}$ and **Ls**, where **U** is a table depicting the use of goods by sectors, **V** a table depicting the outputs of the sectors in terms of goods, **L** the factor input table, and allocation vector $\mathbf{s} \geqq 0$. Similarly, $\sum \mathbf{x}^0_i$ has components **f** and **0**, where **f** is the vector of final goods consumption. The maximal consumption is $c\mathbf{f}$, which we enter in the denominator, as discussed at the end of the last section. The respective prices are denoted **p** and **w**, respectively; these are the shadow prices of program (9). The generalized Solow residual thus becomes

$$SR = -\{\mathbf{p}^{\top} d[(\mathbf{U} - \mathbf{V}^{\top})\mathbf{s}] + \mathbf{w}^{\top} d(\mathbf{L}\mathbf{s})\}/\mathbf{p}\cdot(c\mathbf{f}) \qquad (26)$$

The shadow prices fulfill the dual constraint,

$$\mathbf{p}^{\top}(\mathbf{U} - \mathbf{V}^{\top}) + \mathbf{w}^{\top}\mathbf{L} - \boldsymbol{\sigma}^{\top} = \mathbf{0} \qquad (27)$$

where $\boldsymbol{\sigma}$ is the shadow price of $\mathbf{s} \geqq 0$. The product rule and substitution of equation (27) into expression (26) reduce the generalized Solow residual to

$$SR = -[\mathbf{p}^{\top} d(\mathbf{U} - \mathbf{V}^{\top})\cdot\mathbf{s} + \mathbf{w}^{\top} d\mathbf{L}\cdot\mathbf{s} + \boldsymbol{\sigma}^{\top} d\mathbf{s}]/\mathbf{p}\cdot(c\mathbf{f}) \qquad (28)$$

By the phenomenon of complementary slackness,

$$\boldsymbol{\sigma}\cdot\mathbf{s} = 0 \qquad (29)$$

expression (28) becomes

$$SR = (\mathbf{p}^{\top} d\mathbf{V}^{\top} - \mathbf{p}^{\top} d\mathbf{U} - \mathbf{w}^{\top} d\mathbf{L} + d\boldsymbol{\sigma})(diag\ \mathbf{V}\mathbf{p})^{-1}(diag\ \mathbf{V}\mathbf{p})\mathbf{s}/\mathbf{p}\cdot(c\mathbf{f}) \qquad (30)$$

Here $(\mathbf{p}^\top d\mathbf{V}^\top - \mathbf{p}^\top d\mathbf{U} - \mathbf{w}^\top d\mathbf{L} + d\boldsymbol{\sigma})(diag\ \mathbf{Vp})^{-1}$ is the row vector of sectoral Solow residuals, while $(diag\ \mathbf{Vp})\mathbf{s}/\mathbf{p}\cdot(c\mathbf{f})$ is the vector of Domar weights, which add to the gross output/net output ratio of the economy, a number greater than one.[14]

To include sectoral efficiency changes, recall from section 5 that the optimal (sustaining maximal consumption) and utilizable resources have only factor components, namely **Ls** and **N**, respectively. Application of the phenomenon of complementary slackness and the main theorem of linear programming to (9) yields $\mathbf{w}^\top\mathbf{Ls} = \mathbf{w}\cdot\mathbf{N} = c$. According to the Corollary, ρ^* is the inverse of this expression. It follows that the efficiency term of *TFP* becomes, substituting in the denominator $\mathbf{w}^\top\mathbf{Ls} = \mathbf{p}^\top(\mathbf{V}^\top - \mathbf{U})\mathbf{s} = \mathbf{p}\cdot(c\mathbf{f})$ by equations (27), (29), and material balance (9),[15]

$$d\rho^*/\rho^* = -d(\mathbf{w}^\top\mathbf{Ls})/(\mathbf{w}^\top\mathbf{Ls}) = -\sum_j [d(\mathbf{w}\cdot\mathbf{l}_{.j}s_j)/(\mathbf{p}\cdot\mathbf{v}_{j.}s_j)]\cdot(\mathbf{p}\cdot\mathbf{v}_{j.}s_j)/\mathbf{p}\cdot(c\mathbf{f}) \quad (31)$$

where the summation is over sectors. The efficiency growth is a Domar weighted average of optimal factor input reduction growth rates.

A further specification is that of *input-output analysis*, where **U** and **V** are square matrices, $\mathbf{V}^\top\mathbf{s}$ is denoted **q**, the vector of (optimal) gross outputs, and $\mathbf{A} = \mathbf{U}(\mathbf{V}^\top)^{-1}$ and $\mathbf{F} = \mathbf{L}(\mathbf{V}^\top)^{-1}$ are the matrices of (intermediate and factor) input coefficients. Expression (26) for the generalized Solow residual becomes

$$SR = -\{\mathbf{p}^\top d[(\mathbf{A} - \mathbf{I})\mathbf{q}] + \mathbf{w}\cdot d(\mathbf{Fq})\}/\mathbf{p}\cdot(c\mathbf{f}) \quad (32)$$

and price equation (27) reads

[14] This number is also called the Domar ratio. For any vector **x**, *diag* **x** denotes the diagonal matrix with **x** on the diagonal.

[15] According to program (9), the material balance is an inequality. However, the premultiplication by the price vector eliminates the slack, by the phenomenon of complementary slackness. Alternatively, the material balance may be transformed to an equality in the same way that equation (11) was derived from program (10), assuming free disposal. Vectors $\mathbf{l}_{.j}$ ($\mathbf{v}_{j.}$) denotes the *j*-th column (row) of matrix **L** (**V**).

$$\mathbf{p}^\top(\mathbf{A} - \mathbf{I}) + \mathbf{w}^\top\mathbf{F} - (\mathbf{V}^{-1}\boldsymbol{\sigma})^\top = \mathbf{0} \tag{33}$$

Assume $\mathbf{s} > \mathbf{0}$.[16] As shadow prices are nonnegative, equation (29) sets the last term of equation (33) zero:

$$\mathbf{p}^\top(\mathbf{A} - \mathbf{I}) + \mathbf{w}^\top\mathbf{F} = \mathbf{0} \tag{34}$$

This permits the following rewrite of the generalized Solow residual, (32):

$$SR = -(\mathbf{p}^\top d\mathbf{A} + \mathbf{w}^\top d\mathbf{F})(diag\ \mathbf{p})^{-1}(diag\ \mathbf{p})\mathbf{q}/\mathbf{p}\cdot(c\mathbf{f}) \tag{35}$$

This is essentially formula (12) of Wolff (1994).[17] The first half of this expression, $-(\mathbf{p}^\top d\mathbf{A} + \mathbf{w}^\top d\mathbf{F})(diag\ \mathbf{p})^{-1}$, is the row vector of sectoral Solow residuals and the remainder, $(diag\ \mathbf{p})\mathbf{q}/\mathbf{p}\cdot(c\mathbf{f})$, is the vector of Domar weights, which add to the gross output/net output ratio of the economy, a number greater than one.[18] Expression (35) details the right hand side of equation (21): The generalized Solow residual measures the shift of the production function by means of reductions in intermediate and factor input coefficients.

The inclusion of sectoral efficiency changes is analogous to equation (31), obtained by substitution of $\mathbf{Ls} = \mathbf{Fq}$, equation (34), and material balance $(\mathbf{I} - \mathbf{A})\mathbf{q} = c\mathbf{f}$:[19]

$$d\rho^*/\rho^* = -d(\mathbf{w}^\top\mathbf{Fq})/(\mathbf{w}^\top\mathbf{Fq}) = -d[\mathbf{p}^\top(\mathbf{I} - \mathbf{A})\mathbf{q})]/\mathbf{p}\cdot(c\mathbf{f})$$
$$= -\textstyle\sum_k\{d[(p_k - \mathbf{p}\cdot\mathbf{a}_{\cdot k})q_k]/p_kq_k\}\cdot(p_k\, q_k)/\mathbf{p}\cdot(c\mathbf{f}) \tag{36}$$

[16] Well known sufficient conditions are $\mathbf{f} > \mathbf{0}$ and $\mathbf{A}$ has nonnegative Leontief inverse. For details see ten Raa (1995), chapter 2.
[17] Wolff (1994) substitutes observed values for gross output $\mathbf{q}$ and final goods consumption $c\mathbf{f}$, which are optimal. However, since gross output and final goods consumption are linked by the same Leontief inverse, $\mathbf{q}$ is obtained by inflating observed gross output by c. As this factor cancels against the one in the denominator, the difference is immaterial.
[18] The input-output disaggregation, (35), is slightly different than the activity analytic one, (30), as sectors are now defined in terms of products, but the totals are the same. This wedge disappears when secondary products are absent (in the sense that output table $\mathbf{V}$ is diagonal).
[19] See footnote 13.

where the summation is over commodities. The efficiency growth is a Domar weighted average of optimal factor input or value-added reduction growth rates.

9. Conclusion

In this paper I have interrelated Debreu's coefficient of resource allocation, the Solow residual and total factor productivity growth. Freed from individual data requirements, Debreu's coefficient of resource allocation growth rate and the Solow residual sum to TFP growth. The procedure is equivalent to the imposition of Leontief preferences. The decomposition of TFP growth into the Solow residual and Debreu's term is the neoclassical counterpart to the decomposition of productivity growth into technical change and efficiency change made in frontier analysis and admits breakdowns by factor input as well as by sector.

10. References

Debreu, Gerard "The Coefficient of Resource Utilization," *Econometrica* 19, 3, 273-292 (1951)

Färe, Rolf, and Shawna Grosskopf "Productivity and Intermediate Products: A Frontier Approach," *Economics Letters* 50, 65-70 (1996)

Färe, Rolf, Shawna Grosskopf, M. Norris, and Z. Zhang "Productivity Growth, Technical Progress, and Efficiency Change in Industrialized Countries," *American Economic Review* 84, 1, 66-83 (1994)

Hulten, Charles R. "Growth Accounting with Intermediate Inputs," *Review of Economic Studies* 45, 3, 511-518 (1978)

Jorgenson, Dale W., and Zvi Griliches "The Explanation of Productivity Change," *Review of Economic Studies* 34, 3, 308-350 (1967) ten Raa, Thijs *Linear Analysis of Competitive Economies*, LSE Handbooks in Economics, Harvester Wheatsheaf/Prentice Hall, Hemel Hempstead (1995)

ten Raa, Thijs, and Pierre Mohnen, "Neoclassical Growth Accounting and Frontier Analysis: A Synthesis," *Journal of Productivity Analysis* 18, 11-128 (2002)

Solow, Robert M. "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics* 39, 3, 312-320 (1957)

Wolff, Edward N. "Productivity Measurement within an Input-Output Framework," *Regional Science & Urban Economics* 24, 1, 75-92 (1994)