

Complication and Complexity in Dynamics of Linear Spatial Socio-Economies, A Synopsis.

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I. Introduction: A new Paradigm: Complication as Evolving Complexity.

The purpose of this article is to provide an explanation of the process of complication, i.e. the deepening and evolution of complexity in evolving complex linear systems in Human Geography and Regional Science.

In this article we will apply the paradigm of complexity and complication to the three main branches of Economic Geography and Regional Science connected with linear systems of flows, networks and superposition of their hierarchies: Linear and Transportation Programming optimization problems, Superposition Principle in Push-Pull analysis of Migration Streams, and Central Place theory.

The main feature of the evolution of a complex system is the emergence of new properties which did not exist in previous trends and which add new information to the system (see, for example, Cowan *et al*, 1994). The evolution of complexity (complication) of the physical universe included in the past at least two quintessential events: the Big Bang, which flooded the universe with radiation and, a billion years later, the darkening of the firmament, because of the appearance of atoms and the creation of stars, black holes and galaxies. The Big Bang started the work of the Universal Engine of Complication, i.e., the machine of the deepening and evolvement of complexity. The rate of complication of the physical universe is very low, while the complication of biological, ecological and especially social reality has continued at an accelerated rate.

At the start we should stress the difference between appearance of new information (invention) in the complex socio-spatial system and a spread of this information (innovation diffusion). Spread of information within the complex system presents the essence of the process of complication. This spread shows itself through the partial adoption of new information and manifests itself through the path dependent process of self-organization within socio-spatial complex system.

The innovation diffusion is the universal property of all complex system and the quintessence of the complication process. The detailed description of the innovation diffusion theory can be found in the publications Sonis, 2000, 2001.

The present article considers the forms of self-organization in linear systems. A system called linear if its states satisfied the system of linear constraints.

Geometrically the set of all admissible states of linear system presents itself in the form

of convex polyhedron in a many-dimensional space (see Weyl, 1935). The vertices of this polyhedron are the optimal solutions of the Linear Programming Optimization problem (Dantzig, 1951).

The actual state of the linear system is the point within the convex polyhedron. The actual state belongs also to the set of simplexes generated by some subset of the vertices of polyhedron. So the dynamics of linear system includes the movement of actual state within the convex polyhedron of the admissible states and the catastrophic jumps of the surrounding simplexes. The self-organization of the linear systems in a simplest form appears as the optimization tendencies of different organizations of space and society.

The algorithmic form of such optimization processes in the case of a general Linear Programming Optimization problem and the particular case of the classical Minimal Cost Transportation Problem is treated in the section II. The central point here is the structural stability of optimal solutions within the Cone-Wedge domains of structural stability and structural changes (“catastrophe” effects) of optimal solutions on the boundaries of structural stability.

In this study the classical theory of Central Places is reconstructed on the basis of Barycentric Calculus and the Christaller and Lösch principles Central Place principles presented as principles of optimal organizations of space.

More elaborated form of self-organization of the linear systems is the superposition of different optimization tendencies which acting simultaneously and obtain only partial representation in the socio-spatial system. Their weighted superposition (convex combination) reflects the results of self-organization of society in space. The Rank-Size sequence of these weights represents the hierarchy of extreme tendencies partially represented in the concrete socio-spatial system. The application of this principle for the analysis of non-optimal transportation flows, migration flows and new decomposition models of the Central Place hierarchies is discussed in detail.

II. Catastrophe effects in Linear Programming

II.1. Cone-Wedge presentation of the domain of Structural Stability of optimal solutions.

In this section we consider the simplest form of self-organization of linear systems: the optimization tendencies of different organizations of space and society. We will start with the consideration of the classical Linear Programming optimization problem.

The central point of this section is the structural stability of optimal solutions of the Linear Programming problem within the Cone-Wedge domains of structural stability and structural changes (“catastrophe” effects) of the basis of optimal solutions on the boundaries of structural stability.

The domain of stability of the basis of optimal solutions in linear programming is the

aggregation of three different domains: (1) the domain of permissible changes of the resources (coefficients of the system of linear constraints); (2) the domain of admissible changes in prices (coefficients of the objective function) and (3) the domain of admissible changes in technological coefficients under which the optimal solutions of the linear programming problem will correspond to the same basis, i.e., to the same set of possible components of optimal solutions.

The description of the domain of the basic stability has a deep economic significance since, from an economic viewpoint, the construction of the domain of the basis for stability corresponds to the determination of the permissible levels in the variation of production costs, permissible levels of resource fluctuations and permissible changes in technological coefficients under which the optimal assortment of output is preserved. In essence, these conditions reflect the basis for the preservation of the optimal arrangement of the economic system. or conditions of optimal organization of space in spatial system. The potential link with input-output analysis provides for the intriguing possibilities of exploring ways in which prices (or quantities) can be used as a tool for the optimal management of an economic system undergoing technological changes or for a system of regions facing changing competitive pressures.

This section focuses on the description of the sensitivity analysis of the optimal solutions of the linear programming problem under conditions of unchanging technology. This implies that only the coefficients of the objective function and the right parts of the system of linear inequalities are arbitrarily changing. We chose such form of sensitivity analysis which describe the catastrophe effects in optimal solutions structure. The description of these effects is based on the polyhedral form of general sensitivity analysis for classical Linear Programming problem (see Sonis, 1982): Consider a primal linear programming problem **LP** and its associated dual **D**:

$$\begin{array}{ll} \text{LP:} & \begin{cases} AX = b \\ X \geq 0 \end{cases} \\ & cx \rightarrow \min \end{array} \quad \begin{array}{l} \text{D:} \\ Yb \rightarrow \max \end{array} \quad YA \leq c \quad (\text{II.1})$$

Let A_0 be an invertible submatrix of the matrix A with the inverse $A_0^{-1} = B$ with the properties:

$$B \geq 0, \quad c_0 B A \leq c \quad (\text{II.2})$$

where the coordinates of the vector c_0 correspond to the columns of the matrix A_0 . Then the primal problem has the optimal solution, X , with the vector of non-zero basis

components X_0

$$X_0 = Bb \quad (\text{II.3})$$

and the dual has the optimal solution:

$$Y = c_0 B \quad (\text{II.4})$$

This proposition also provides the complete description of the domains of the structural stability of the optimal solutions for the primal and dual linear programming problems under conditions of unchanging technology.

If the resources, b , and prices, c , are changed, a polyhedral *cone* in the spaces of resources:

$$C = (b : Bb \geq 0) \quad (\text{II.5})$$

and a polyhedral *wedge* in the space of costs:

$$W = (c : c_0 BA \leq c) \quad (\text{II.6})$$

are obtained. These determine the domains of stability of the basis for the optimal solutions of the primal and dual problems. Thus, the Cartesian product $C \times W$ defines the domains of the structural stability of the optimal solutions for the primal and dual problem. The construction of the Cartesian product for each given optimal solution is simple, because the last tableau of the simplex algorithm of Dantzig, 1963, contains the components of the matrix BA . Hence, to obtain the inequalities determining the domain of the structural stability, access to the components of the last simplex tableau will suffice.

Moreover, the optimal solutions, X and Y , associated with the basis matrix A_0 are the extreme points or vertices of the corresponding convex polyhedrons of the admissible solutions for the primal and dual problems. Since the matrix A contains a finite number of invertible submatrices, the space of resources and the space of costs are decomposable into a finite number of domains:

$$C_1 \times W_1, C_2 \times W_2, \dots, C_r \times W_r \quad (\text{II.7})$$

so that each of them corresponds to the preservation of some invertible basis submatrix of the matrix A , i.e., to the preservation of some optimal assortment of production.

The transition from the domain $C_i \times W_i$ to the next domain $C_{i+1} \times W_{i+1}$ may be described as the intersection of one of the bounds of the cone C_i or wedge W_i . In this case, outside

the cone, C_i , the criterion of optimality will fail to hold in the cell of the objective row of the simplex tableau corresponding to the bound of the transition. This cell defines the type of production to be introduced into the basis and to construct the cone C_{i+1} , only one step of the simplex algorithm is needed.

If the transition through the bound of the wedge, W_i , takes place, then the condition of positivity of the components of the optimal solution fails to hold in the row corresponding to the chosen bound. This bound defines the type of production to be eliminated from the basis of the solution and, as before, only one further step is necessary in the dual simplex algorithm. Hence, by means of a sequential transition from domain to domain, the full sensitivity of the optimal solutions of the linear programming problem under conditions of unchanging technology can be revealed.

It should be noted that the description of the domains of the basis stability solves the multivariant analysis problem by reducing this problem to a relatively small number of variants. This reduction may be carried out in the following way. From the collection of all variants of prices and resources, select an arbitrary variant and solve the corresponding linear programming problem and construct the domain of the basis stability for the optimal solution. Further, from the collection of the variants, select those corresponding to the same domain of the basis stability. The remaining variants are treated similarly.

III. Structure of optimal (minimal cost) transportation flows.

The actual hierarchy of urban settlements puts strong restrictions on the spatial organization of optimal (minimal cost) transportation flows between the settlements. In turn, the spatial and temporal stability of the transportation flows may be essential factor of growth or decline of a hierarchy of urban settlements. Usually the optimal transportation flow does not cover all linkages of the transportation network between the settlements; therefore the existence of structurally stable optimal transportation flows can result in a change of the transportation network itself on the expense of the non-used linkages.

III.1. Arc-density property of optimal networks.

Consider the cost minimization problem on a network with m suppliers with a_i units of supply for each of i suppliers ($i=1, \dots, m$) and n demanders with corresponding needs for b_j units ($j=1, \dots, n$) and with the conservation condition that $\sum a_i = \sum b_j$. Given a set of costs, c_{ij} , for each supply-demand link, the optimal solution will occupy $m+n-1$ cells (the basis cells) of the computing table. Topologically, the set of basic cells

defines the maximally connected subgraph of the transportation network without cycles and includes exactly $m+n-1$ arcs. For each non-basic cell, the cycle exists whose vertices (excluding the non-basic cell itself) are basic cells.

Consider a connected planar graph with v vertices and a arcs. Each vertex can be origin (for supply) or destination (for demand) or origin/destination (for supply and demand simultaneously). A question arises: whether or not this planar graph can represent the topological structure of optimal transportation network for some minimum cost transportation problem? The answer is based on the following

Arc-density theorem (Sonis, 1982): If the connected planar graph with v vertices and a arcs represents a topological structure of the minimal cost transportation flow of homogeneous production, then its arc density $(a+1)/v$ has a following range:

$$1 \leq \frac{a+1}{v} \leq 2 \quad (\text{III.1})$$

III.2. Domains of structural stability and boundaries of structural change in optimal transportation networks.

Consider the cost minimization problem on a network with m suppliers with a_i units of supply for each suppliers ($i=1, \dots, m$) and n demanders with corresponding needs for b_j

units ($j=1, \dots, n$) such that the total supply is equal to total demand: $\sum a_i = \sum b_j$, and

let c_{ij} be the cost of transportation of one unit of production from i th supplier to j th demander.

The description of the domains of the basis stability provide the mechanisms for finding the optimal linkages between demanders and suppliers; the difficulty here is that the solution to the transportation problem does not provide the last simplex tableau and this must be restored. For the re-establishment of the matrices, BA and B a generalization of the MODI method is used (Dantzig, 1951) providing a connection with the simple structure of the matrix associated with the transportation problem.

The Vector Method of Potentials and matrix inequalities of Cone-Wedge domains of structural stability of optimal networks can be find in Sonis, 1982a.

III.2.1. Structural change in the spatial structure of optimal transportation flows

The change in the spatial structure of the optimal transportation flow is connected to the absence of fulfillment of one or more of the inequalities defining the cone and wedge of the structural stability. The domains of the structural change are the faces of

the domain of the structural stability $C \times W$, which are the closed hyperplanes in the supply-demand space or in the space of transportation costs. On the face of the cone C , the flow is degenerated; it divides into a few independent subflows that are the optimal solutions a smaller size problem. If one moves out of the cone C , then the admissible flow with a given topological structure does not exist and a new flow must be constructed. If one moves out of the wedge W , then there is an admissible flow with a previous topological structure, but the condition of optimality of the transportation flow fails to hold, and the structure of the flow must be changed by substituting one arc of the spatial structure for another.

III.3. Behavioral competition between suppliers and demanders within the minimal cost transportation problem.

In this section, it is shown that in the minimum cost linear programming transportation problem, the *global collective* minimization of cost implies a totally antagonistic competitive exclusion behavior on the part of suppliers and demanders. This principle will surface again in the application of the superposition ideas.

III.3.1. Competitive exclusion behavioral rules in the minimum cost solution.

It is well known in the linear programming transportation problem that the competitive forces that result in an optimal allocation may lead to the exclusion of some subgroups; this effect will now be explored in the form of behavioral rules for subsets of suppliers and demanders. Now consider an arbitrary subset of all the basic cells; the suppliers and demanders in this subset will be referred to as the *old* suppliers and demanders and the complement set will be referred to as the *new* suppliers and demanders. The following three rules comprise the competitive exclusion effect (see Sonis, 1993):

1. each *new* demander can be served by only one *old* supplier;
2. each *new* supplier can serve only one *old* demander;
3. if a *new* demander is served by both *old* and *new* suppliers, then this *new* supplier cannot serve any other *old* demander.

Thus, in minimal cost transportation problem the *global collective minimization of costs* implies the totally antagonistic *competitive exclusion individual behavior* of suppliers and demanders.

These behavioral rules allow constructing the geometric and numeric algorithm of enumeration of all basic subgraphs presenting spatial structure of the transportation network carrying the optimal transportation flows under various requirement on

supply-demand and transportation costs.

The following question arises: what spatial form has any admissible basis subgraph in the hexagonal network? The behavioral rules presented in the previous subsection allow the enumeration of all basic subgraphs presenting spatial structure of the optimal transportation flows in hexagonal network under various requirements on supply-demand and transportation costs.

IV. Superposition Principle – the inverted problem of Multi-objective Programming.

IV.1. Connection between the Weber principle of industrial location and the Moebius Barycentric Calculus.

Geometrically, the solution of the Linear Programming optimization problem is taking into account only one vertex of the convex polyhedron of all admissible solutions. The information about the set of all vertices and the structure of the convex polyhedron, while it is important for deriving the solution, is neglected in the solution itself. Moreover, the actual state of the linear regional system (a system defined by linear balancing constraints) is usually far from whatever optimization. From the view-point of optimization the actual state of a regional system is a solution for an optimization problem of multi-objective programming. This means that the actual state reflects the existence of a set of different extreme tendencies or trends corresponding to the optimization of a set of different objective functions. But simultaneous optimization of two or more objective functions is inaccessible mathematically (Boltiansky, 1973, paragraph 1.5). Therefore, the problem of multi-objective programming is usually transformed to the problem with only one objective. Traditionally there are two approaches for this transformation (Cohon, 1978). One of them is to optimize one of objectives while appending the other objectives to a constraint set, so that the (sub-optimal) solution would satisfy these objectives up to an acceptable level. The other approach is to optimize a super-objective function created by weighted sum of a set of objectives. There is a great deal of arbitrariness in both approaches and the influence of each objective is distorted; therefore, the optimal solution of the multi-objective programming is usually removed from the actual state of the regional system.

The problem became much easier if we replace the consideration of multi-objective optimization with the problem of analysis of an actual state of linear regional system. Geometrically, the actual state belongs to the convex polyhedron of admissible solutions; the vertices of this polyhedron are the optimal solutions of one objective optimization problems. So we find ourselves in the typical situation of the theory of convex polyhedrons: a point (of actual state) within the convex polyhedron (of admissible

solutions). The central fact of the theory of convex polyhedrons is the Minkovski, 1910, theorem about the center of gravity of a convex polyhedron: it is possible to hang the collection of weights (with common weight 1) on the vertices of the convex polyhedron such that its center of gravity will coincide with a given point. More precisely, the Minkovski theorem can be formulized in the following manner: every point Y_1 of a convex bounded many-dimensional polyhedron can be presented as a convex combination (a weighted sum) of several vertices X_1, X_2, \dots, X_k :

$$Y_1 = p_1 X_1 + p_2 X_2 + \dots + p_k X_k, \quad 0 \leq p_i \leq 1, \quad i = 1, 2, \dots, k, \quad p_1 + p_2 + \dots + p_k = 1 \quad (\text{IV.1})$$

The Minkovski theorem can be interpreted as an inversion of the classical A. Weber principle of industrial location (Weber, 1909). Weber's main idea was the utilization of the notion of center of gravity: the optimal location of a plant is the center of gravity of a polygon whose vertices correspond to the location of raw materials, energy man-power and the market location. We shall use the following inversion of Weber's principle: the point of the actual state of the regional system is considered as a center of gravity of the polyhedron of admissible states of the regional system. So we determine the collection of vertices X_i and their weights (baricentric coordinates) p_i such that the center of gravity of the polyhedron of admissible states will coincide with the actual state. Thus, the problem of analysis of an actual state of the regional system is reduced to the basic problem of Barycentric Calculus (Möbius, 1827).

IV.2. The Caratheodory theorem and the inverted problem of multi-objective programming.

The important specification of the Minkovsky theorem is the Caratheodory, 1911, theorem: every point Y_1 within a convex closed bounded n-dimensional polyhedron can be presented by a convex combination of vertices, X_1, X_2, \dots, X_{m+1} , belonging to some m-dimensional simplex ($m \leq n$) with m+1 vertices:

$$Y_1 = p_1 X_1 + p_2 X_2 + \dots + p_{m+1} X_{m+1}, \quad 0 \leq p_i \leq 1, \quad i = 1, 2, \dots, m+1, \quad p_1 + p_2 + \dots + p_{m+1} = 1 \quad (\text{IV.2})$$

In other words, the given point Y_1 is a center of gravity of the set of weights p_1, p_2, \dots, p_{m+1} hanging on the vertices of certain simplex. Moreover, the barycentric coordinates p_1, p_2, \dots, p_{m+1} of Y_1 with respect to a fixed simplex are defined uniquely.

This theorem plays only auxiliary role in the linear optimization theory; in our article it will be the base of the superposition principle of our linear regional analysis: each actual state of the linear regional system is the superposition of a set of extreme states of the regional system, which are the optimal solutions of the sequence of optimization problems, presenting the simultaneous action of different extreme tendencies within regional system. The weights (barycentric coordinates) of the extreme states define the measure of their realization in the actual state.

In the case of a linear regional model given by the system of linear constraints the superposition principle can be presented as the inverted problem of multi-objective programming:

Let Y_1 be an admissible solution of the system of linear constraints:

$$\begin{cases} AX = b \\ X \geq 0 \end{cases} \quad (\text{IV.3})$$

and let

$$f_1(X), f_2(X), \dots, f_s(X) \quad (\text{IV.4})$$

be the ordered set of linear or concave objective functions. Then there is the decomposition of Y_1 into convex combination

$$Y_1 = p_1 X_1 + p_2 X_2 + \dots + p_s X_s + p_{s+1} Y_{s+1}, \quad 0 \leq p_i \leq 1, \quad i = 1, 2, \dots, s+1, \quad p_1 + p_2 + \dots + p_{s+1} = 1 \quad (\text{IV.5})$$

where Y_{s+1} is the unexplored remainder state and each vector X_i is the optimal solution to the optimization problem:

$$\begin{aligned} & \max f_i(x) \\ & \text{subject to constraints: } \begin{cases} AX = b \\ X \geq 0 \end{cases} \end{aligned} \quad (\text{IV.6})$$

with additional constraints on coordinates of vector X :

$$x_{k_1} = x_{k_2} = \dots = x_{k_{i-1}} = 0$$

The additional zero constraints correspond to the regional “bottle-necks”, i.e., the parts of the regional system where the competition and conflict between different extreme tendencies obtain the most noticeable form. The ordered set of objective functions (IV.4), corresponding to the sequence of extreme tendencies, defines the simplex including the actual state Y_1 . Thus, the decomposition (IV.5) takes into the consideration of the shares of certain extreme tendencies. So obtaining the decomposition we analyze the actual state from the certain preset viewpoint of investigator-analyst. The proof of the decomposition theorem will be presented below in the form of algorithm of decomposition.

IV.3.1. Special case of one linear objective function.

In the case of one linear objective function

$$f_1(X) = f_2(X) = \dots = f_s(X) \equiv f(X) \quad (\text{IV.7})$$

a numerical procedure of the decomposition can be simplified if we take into consideration the fact that points from k -dimensional face include $n-k$ zero coordinates. Therefore, the choice of consequent extreme states X_1, X_2, X_3, \dots can be made with the

help of the same objective function $f(X)$ if we replace in this function the coefficients of variables corresponding to zero coordinates in $X_j, j=2,3,\dots$ by a very large number M and solve the M -problem (the linear programming problem with artificial basis, Dantzig, 1963) with the same system of linear constraints (IV.3).

V. Polyhedral Catastrophic Dynamics of the Push-Pull states of migration streams.

V.1. Description and spatial interpretation of the decomposition procedure.

This chapter deals with an analysis and geographical representation of attraction (Pull) and repulsion (Push) in a real migration stream. At first we restrict ourselves to detailed representation of the Push analysis, since the scheme of Pull analysis can be considered analogously. (For the simplicity we will consider the migration of the homogeneous population of migrants moving within and between the same set of origins/destinations; the consideration of different sets of origin and destinations and the cases of differentiation of migrants by age, sex, nationality, labor specialization, level of education, etc. can be found in Sonis, 1980.)

The Push/Pull analysis requires the following information:

1. A geographical map of the migration origin/destination regions:
2. The choice of a homogeneous migrant population moving during fixed time interval from origins to destinations. This population is statistically described by the origin-destination matrix

$$M = [m_{ij}] \quad (V.1)$$

where $m_{ij} \geq 0, i, j=1,2,\dots,n$ is the number of migrants moving from origin i to destination j .

3. A geographical distribution of the migration "bottle necks", i.e., a list of zero components of the matrix M ;

$$m_{i_1j_1} = m_{i_2j_2} = \dots = m_{i_rj_r} \equiv 0 \quad (V.2)$$

4. An initial distribution of migrants in regions of origin (for Push analysis):

$$N_i = \sum_{j=1}^n m_{ij}, \quad i=1,2,\dots,n \quad (V.3)$$

5. A final distribution of migrants in regions of destination (for Pull analysis):

$$K_j = \sum_{i=1}^n m_{ij}, \quad j=1,2,\dots,n \quad (\text{V.4})$$

This data allows to incorporate the real state of the migratory system M into the polyhedrons of admissible states. For the Push analysis the convex polyhedron of admissible states includes the migration matrices $X = [x_{ij}]$, satisfying a following system of linear constraints:

$$\begin{cases} x_{ij} \geq 0, & i, j=1,2,\dots,n \\ x_{i_1j_1} = x_{i_2j_2} = \dots = x_{i_rj_r} \equiv 0 \\ \sum_{j=1}^n x_{ij} = N_i, & i=1,2,\dots,n \end{cases} \quad (\text{V.5})$$

For the Pull analysis the convex polyhedron of admissible states includes the migration matrices $X = [x_{ij}]$, satisfying a following system of linear constraints:

$$\begin{cases} x_{ij} \geq 0, & i, j=1,2,\dots,n \\ x_{i_1j_1} = x_{i_2j_2} = \dots = x_{i_rj_r} \equiv 0 \\ \sum_{i=1}^n x_{ij} = K_j, & j=1,2,\dots,n \end{cases} \quad (\text{V.6})$$

The polyhedrons (V.5,6) are bounded and lying within many-dimensional rectangular parallelepipeds $x_{ij} \leq N_i(K_j)$. The vertices of these parallelepipeds are defined by the

rule: "everything or nothing"- their coordinates equal either to zero or to $N_i(K_j)$. This rule have the following geographical meaning (Nystien and Dacey, 1961): the extreme tendency represents the repulsion or attraction of migrants only to region to which the largest number of actual migrants are pushed or attracted.

The superposition approach means the decomposition of the migration origin-destination matrix M into the weighted sum of basis matrices M_k representing the action of the extreme tendencies:

$$M = p_1M_1 + p_2M_2 + \dots + p_mM_m \quad (\text{V.7})$$

where $1 \geq p_s \geq 0$ and $p_1 + p_2 + \dots + p_m = 1$.

We interpret this decomposition as a display of the principle of intervening opportunities and competition (Stoufer, 1960): the migrant sees the set of opportunities and selects an

opportunity in the attempt to optimize his own objective. The exchange of the information between the prospective migrants about different opportunities resulted in the spatial migration empirical regularity (Lee, 1966): “a migration tends to take largely within well defined streams” representing different extreme tendencies. The complete expressions of these extreme tendencies define the assemblage of basis matrices M_s . Each extreme flow M_s enters the real flow M with the weight $p_s \leq 1$, and the sum of weights is equal to 1.

The procedure of the Push-Pull migration analysis based on the results of chapter IV.3 consists of the successive extraction from an actual migration stream of the shares corresponding to the constructed set of extreme tendencies. At the beginning we choose the main extreme tendency; then we construct an extreme migration flow, which is the complete expression of this tendency, and determine its share (weight) in the actual migration and simultaneously determine the residual of the actual migration after the extraction of the action of the main extreme tendency. In this residual we choose the next extreme tendency, and so forth. The most significant fact is that the set of residuals corresponds to the migrationally meaningful set of the “bottle necks”, corresponding to that parts of the actual migration where the action of migration factors compels the actual migration to diverge from extreme flow. The appearance of obstacles preventing or supporting the repulsion or attraction from or to some region can be interpreted as the realization of the Stouffer principle of intervening opportunities (Stouffer, 1960). Simultaneously, these migration “bottle necks” determine the weights. The schemes of Push and Pull analysis include the similar numerical procedures. Since “each main current of migration produces a compensating counter-current” (Ravenstein, 1885) the result of Push and Pull analysis usually complemented each other.

The relationship between push and pull migration phenomena obtained in the theoretical migration literature in the “Laws of migration by Ravenstein (Ravenstein, 1885; see also Lee, 1966) the form of the concept of stream and counterstream: “for every major migration stream a counterstream develops”, and “The efficiency of stream and counterstream (i.e. ratio of stream to counterstream or the net migration generated by the opposite flows) tends to be low if origin and destination are similar.

V.2. Polyhedral catastrophic dynamics.

If some temporal sequence of migration origin-destination matrices exists for the sequence of different time periods but the same territorial differentiation of migration origin and destinations, then the corresponding normalized spaces of pull (or push) admissible migration states are coincides with the same unit cube (V.8) (or (V.9)). So, the temporal sequence of migration origin-destination matrices generates the movement of

the point of the normalized pull (push) migration state $R_1 (S_1)$ within the cube of admissible states, and the decomposition of the normalized state generates the simplex whose vertices present the extreme tendencies in the normalized state. So, the temporal sequence of the migration matrices generated the sequence of simplexes, whose vertices belong to the unit cube of normalized admissible states. This temporal polyhedral dynamics is structurally stable, if the sequence includes only identical simplexes. The dynamics is partially structurally stable if the simplexes include the same partial set of identical vertices, presenting the same set of extreme tendencies. The dynamics is catastrophic if there are no identical subsets of vertices.

As an example, let us consider the pull polyhedral catastrophic dynamics of the internal migration of Israeli population during the decade 1985-1994:

$$\begin{aligned}
 1985 \Rightarrow & \begin{bmatrix} 0 & 0.507 & 0.255 \\ 0.753 & 0 & 0.745 \\ 0.247 & 0.493 & 0 \end{bmatrix} = 0.507 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.248 \begin{bmatrix} 0 & \bullet & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + 0.238 \begin{bmatrix} 0 & \bullet & 0 \\ 1 & 0 & 1 \\ \bullet & 1 & 0 \end{bmatrix} + 0.008 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & \bullet \\ \bullet & 1 & 0 \end{bmatrix} \\
 1986 \Rightarrow & \begin{bmatrix} 0 & 0.508 & 0.269 \\ 0.743 & 0 & 0.731 \\ 0.257 & 0.492 & 0 \end{bmatrix} = 0.508 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.249 \begin{bmatrix} 0 & \bullet & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + 0.232 \begin{bmatrix} 0 & \bullet & 0 \\ 1 & 0 & 1 \\ \bullet & 1 & 0 \end{bmatrix} + 0.011 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & \bullet \\ \bullet & 1 & 0 \end{bmatrix} \\
 1987 \Rightarrow & \begin{bmatrix} 0 & 0.504 & 0.255 \\ 0.750 & 0 & 0.745 \\ 0.250 & 0.496 & 0 \end{bmatrix} = 0.504 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.250 \begin{bmatrix} 0 & \bullet & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + 0.241 \begin{bmatrix} 0 & \bullet & 0 \\ 1 & 0 & 1 \\ \bullet & 1 & 0 \end{bmatrix} + 0.005 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & \bullet \\ \bullet & 1 & 0 \end{bmatrix} \\
 1988 \Rightarrow & \begin{bmatrix} 0 & 0.498 & 0.213 \\ 0.761 & 0 & 0.787 \\ 0.239 & 0.502 & 0 \end{bmatrix} = 0.502 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 0.254 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & \bullet & 0 \end{bmatrix} + 0.212 \begin{bmatrix} 0 & 1 & 1 \\ \bullet & 0 & 0 \\ 1 & \bullet & 0 \end{bmatrix} + 0.032 \begin{bmatrix} 0 & 1 & \bullet \\ \bullet & 0 & 1 \\ 1 & \bullet & 0 \end{bmatrix} \\
 1989 \Rightarrow & \begin{bmatrix} 0 & 0.483 & 0.243 \\ 0.770 & 0 & 0.757 \\ 0.230 & 0.517 & 0 \end{bmatrix} = 0.517 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 0.243 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \bullet & 0 \end{bmatrix} + 0.230 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \bullet & 0 \end{bmatrix} + 0.010 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \bullet & \bullet & 0 \end{bmatrix} \\
 1990 \Rightarrow & \begin{bmatrix} 0 & 0.534 & 0.199 \\ 0.824 & 0 & 0.801 \\ 0.176 & 0.466 & 0 \end{bmatrix} = 0.534 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.267 \begin{bmatrix} 0 & \bullet & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 0.173 \begin{bmatrix} 0 & \bullet & 1 \\ 0 & 0 & \bullet \\ 1 & 1 & 0 \end{bmatrix} + 0.026 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & \bullet \\ \bullet & 1 & 0 \end{bmatrix} \\
 1991 \Rightarrow & \begin{bmatrix} 0 & 0.583 & 0.211 \\ 0.846 & 0 & 0.789 \\ 0.154 & 0.417 & 0 \end{bmatrix} = 0.583 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.211 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 0.154 \begin{bmatrix} 0 & \bullet & \bullet \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + 0.052 \begin{bmatrix} 0 & \bullet & \bullet \\ 1 & 0 & 1 \\ \bullet & 1 & 0 \end{bmatrix} \\
 1992 \Rightarrow & \begin{bmatrix} 0 & 0.592 & 0.223 \\ 0.811 & 0 & 0.777 \\ 0.189 & 0.408 & 0 \end{bmatrix} = 0.592 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.219 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 0.185 \begin{bmatrix} 0 & \bullet & 0 \\ \bullet & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + 0.004 \begin{bmatrix} 0 & \bullet & 1 \\ \bullet & 0 & \bullet \\ 1 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
1993 \Rightarrow \begin{bmatrix} 0 & 0.536 & 0.163 \\ 0.840 & 0 & 0.837 \\ 0.160 & 0.464 & 0 \end{bmatrix} &= 0.536 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.301 \begin{bmatrix} 0 & \bullet & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 0.160 \begin{bmatrix} 0 & \bullet & 1 \\ 0 & 0 & \bullet \\ 1 & 1 & 0 \end{bmatrix} + 0.003 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & \bullet \\ \bullet & 1 & 0 \end{bmatrix} \\
1994 \Rightarrow \begin{bmatrix} 0 & 0.500 & 0.185 \\ 0.803 & 0 & 0.815 \\ 0.197 & 0.500 & 0 \end{bmatrix} &= 0.500 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.303 \begin{bmatrix} 0 & \bullet & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 0.185 \begin{bmatrix} 0 & \bullet & 1 \\ \bullet & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + 0.012 \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}
\end{aligned}$$

This sequence of pull decompositions includes three years, 1985-1987, of complete structural stability; i.e., in this time interval all extreme tendencies are repeated and, moreover, their weights (barycentric coordinates) preserved their rank-size ordering. Nevertheless, the places of “bottle neck” problems are stable only partially. In years 1988-1989 the main extreme tendency which is stable in the previous three years are replaced by extreme tendency which was only third in the previous three decompositions, and the main extreme tendency in previous three years became the second in the next two years. The decomposition simplex which was stable in 1985-1987 is replaced in 1988-1989 by decomposition simplex including as vertices the previous main extreme tendencies. In years 1990-1994 the structural stability of pull decomposition is only partial; the decomposition simplexes in these years includes the same main extreme tendency as in 1985-1987. Other tendencies and the corresponding “bottle necks” undergo the different catastrophic changes.

The consideration of the average 1985-1994 pull decomposition

$$\begin{aligned}
1985-1994 \Rightarrow \begin{bmatrix} 0 & 0.524 & 0.222 \\ 0.790 & 0 & 0.778 \\ 0.210 & 0.476 & 0 \end{bmatrix} &= \\
&= 0.524 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 0.254 \begin{bmatrix} 0 & \bullet & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 0.210 \begin{bmatrix} 0 & \bullet & 1 \\ 0 & 0 & \bullet \\ 1 & 1 & 0 \end{bmatrix} + 0.012 \begin{bmatrix} 0 & \bullet & 1 \\ 1 & 0 & \bullet \\ \bullet & 1 & 0 \end{bmatrix}
\end{aligned}$$

shows that the ten years polyhedral catastrophic dynamics represents the oscillation of the simplexes of the actual normalized pull migration states near the simplex of the average normalized pull migration state.

VI. Reconstruction of Central Places Geometry on the basis of Barycentric Calculus.

VI.1. Main assumptions of the Christaller, Losch and Beckmann-McPherson models in the classical theory of the Central Place systems.

The Central Place theory of Christaller and Losch has been in existence more than five decades (Christaller, 1933, and Losch, 1940). Although at present

there is no doubt about the conceptual usefulness of the Central Place theory, its essential deficiency relates to its applicability to the analysis of an actual central place system. Moreover the classical Central Place theory represents the challenge to the New Urban Economics and New Economic Geography which both fail to reproduce and incorporate the spatial basis of the classical theory (*cf.* David, 1999). In this chapter we try to close the existing gap between the pure theoretical Christaller and Lösch models and the structure of an actual central place system; we propose an alternative hierarchical model based on the idea of mixed hierarchy of the Central Place system (Christaller, 1950, p.12; Woldenberg, 1968) and on the Beckmann-McPherson model of Central Place system (Beckmann, McPherson, 1970), which are the intermediate links between the Christaller and Lösch models.

Our scheme of analysis is based on the concept of the center of gravity and its barycentric coordinates in a plane and within the convex polyhedron (simplex) in multi-dimensional space.

It is interesting to note that the barycentric coordinates appeared in a latent and mysterious form in the geometry of the Central Place theory – in the form of the rhombic coordinates x and y in the primary Christaller triangular lattice (Dacey, 1964, 1965) or in the form of the Tinkler, 1978, coordinate triples $(x, y, x+y)$, where x, y are the rhombic coordinates. Neither Dacey nor Tinkler realized that the triple (x, y, z) where $z = 1 - x - y$ present three barycentric coordinates in a plane.

The introduction of the barycentric coordinates essentially simplifies the geometry of the Central Place theory. Moreover, the imbedding of an actual central place system into the many-dimensional polyhedron of all possible states of the central place system and the evaluation of the barycentric coordinates

for an actual central place system allows the application of the Superposition Principle presented earlier in the Chapter III. The Superposition principle allows the construction of a model of the hierarchical structure of an actual state of a central place system as a convex combination of the Beckman- McPherson models, which are the “building blocks”- extreme states of the central place system.

Before the presentation of this hierarchical decomposition model, it might be useful to present briefly the ideas of Christaller, 1933, 1950, Lösch, 1940,

Beckman , Mc-Pherson, 1970, Woldenberg, 1968, 1979, Parr, 1978a,b, 1981 and Sonis, 1985, 1986. All these ideas are concerning the concept of mixed central place hierarchies.

The spatial description of the original Christaller Central Place model is based on three generic geometric properties of central places associated this Central Place system:

1. The first property is that all hinterland areas of the central places at the same hierarchical level form a hexagonal covering of the plane with the centers on the homogeneous triangular lattice presenting the centers of the hexagons from the Christaller primary covering.
2. The second property is that the size of the hinterland areas increases from the smallest (on the lower tier of Central Place hierarchy) to the largest (on the highest tier of hierarchy) by a constant nesting factor k .
3. The third property is that the center of a hinterland area of a given size is also the center of an hinterland of each smaller size (Christaller, 1933).

By definition the nesting factor is the ratio between the areas S of the hexagons belonging to some hexagonal covering of the plane to the area s of hexagons belonging to the primary Christaller covering by smallest hexagons with the property: the distance between the centers of smallest hexagons equals 1:

$$k = \frac{S}{s} \quad (\text{VI.1})$$

It is easy to see that if d is the distance between the centers of adjacent hexagons of some hexagonal covering of the plane then the area of each hexagon is equal to $S = 2\sqrt{3}d^2$, so the area of smallest hexagon from the Christaller primary covering equal $s = 2\sqrt{3}$; thus, the nesting factor equals to the square of the distance between the centers of adjacent hexagons of hexagonal covering of the plane:

$$k = d^2 \quad (\text{VI.2})$$

4. The nesting factors 3,4,7 play the most important role in the Christaller Central Place theory: they express one of the Christaller's three principles, namely, marketing ($k = 3$), transportation ($k = 4$) and administrative ($k = 7$) principles. The nesting factors 3,4,7 generate three geometrical sequences of the hexagonal market area sizes: 1,3,9,27,..., 3^n , ...; 1,4,16,64,..., 4^n , ...;

1,7,49,343,...,7ⁿ,..., As an example the three-tier Christaller Central Place hierarchies are represented on the figure VI.2.

<Figure VI.2. Three-tier Christaller Central Place hierarchies corresponding to the sequences of the nesting factors 1,3,9; 1,4,16 and 1,7,49.>

It is possible to interpret these Christaller principles as principles of optimal organization of central place market areas: marketing principle represents the minimal number of small market areas (3) included in a bigger market area; the transportation principle present such optimal organization of space where the transportation network between two bigger central places passes through the smaller central place; the administrative principle presents such optimal organization of space where the administrative hinterland of the larger central place includes almost completely the set of administrative hinterlands of smaller central places.

Christaller, 1950, himself came to realize that the marketing, transportation and administration principles could be expected to act simultaneously in geographical space. He suggested modifying his original model by a mixing of the nesting factors 3,4,7 into the grouping non-integer nesting factor $k = 3.3$ which generates the geometric progression 1,3.3,10,33...

Woldenberg, 1968, elaborated on analogy between the hierarchical structure of fluvial systems and the hierarchical structure of the hinterlands of the central place systems, so as to be able to generate the sequences of nesting factors for sizes of market areas for central place systems. With the help of numerical computer model Woldenberg, 1979, compared the results of computer simulations with a wide set of actual central place hierarchies and mentioned certain difficulties that rise in attempting to describe an actual hierarchy in terms of the numerical computer model. The weak points of these generic models are the non-uniqueness of the procedure of grouping and an empirism in the underlying theoretical reasoning.

The L schian hexagonal landscape (L sch, 1940) is the superposition of all possible coverings of a plane by hexagons whose centers are coincide with the vertices of the triangular lattice and the sizes of market areas are integers. The L schian model defines a set of the Kanzig-Dacey nesting factors (Dacey, 1964):

$$k = x^2 + y^2 + xy \quad (\text{VI.3})$$

where x, y are arbitrary integers (rhombic or barycentric coordinates of the central places), so nesting factors in the Löschian landscape are

$$k = 1, 3, 4, 7, 9, 12, 13, 16, 19, \dots \quad (\text{VI.4})$$

The geometric procedure for construction of the Löschian landscape is simple and straightforward: for the derivation of a part of the Löschian landscape which corresponds to the hexagonal covering with a nesting factor $k = d^2$ one should choose on the Christaller primary lattice two points with the distance d between them, to derive the segment connected these two centers and from its middle point to draw a perpendicular segment of the size $\frac{d}{\sqrt{3}}$. The end point of this

perpendicular segment is the vertex of the hexagon and, thus defines the position of whole hexagon and all hexagons from the corresponding coverings. Lösch himself constructed the coverings corresponding to 150 nesting factors.

Parr indicated (Parr, 1970, p.45) that these Löschian landscape nesting factors also present the optimal organizations of space similar to Christaller marketing, transportation and administrative principle; for example, the nesting factors 13 and 19 have the same property of administrative convenience as factor 7, while factors 9 and 16 have the same transportation efficiency as factor 4. According Lloyd and Dicken, 1972, p. 49, “Lösch suggested that this spatial arrangement of urban centers was consistent with what he saw to be a basic element in human organization: the *principle of least effort*.”

The Beckmann-McPherson, 1970, Central Place model differs from the Christaller framework by applying variable nesting factors and by using the principle of possible coverings of the plane by hexagons of variable integer sizes. Their centers are the vertices of the initial Christaller triangular lattice. The Christaller model is only a partial case of Beckmann-McPherson models. Simultaneously, the Beckmann-McPherson models are an incomplete case of the Löschian model – incomplete in the sense that the Beckmann-McPherson models include only a small part of the hinterland areas from the Löschian landscape. In this chapter we will consider a set of Beckmann-McPherson Central place systems including a single largest central place and the finite number of hierarchical levels. So, the Beckmann-McPherson model is defined with the help of the sequence of the Kanzig-Dacey nesting factors (VI.2) k_1, k_2, \dots, k_n representing the nesting properties of the consecutive hierarchical levels.

The existence of variable nesting factors on different hierarchical levels of the Beckmann-McPherson model represents the simultaneous action of the Christaller marketing, transportation and administrative principles and the corresponding Lösschian optimization principles.

Parr, 1970, described the way to compare the theoretical models with the structure of the actual central place system. His idea was to use the Beckmann-McPherson Central place model as the best fitting approximation of an actual central place hierarchy. Parr also met with difficulties which arise from the omission of the analysis of the discrepancy between the actual central place hierarchy and its best fitting Beckmann-McPherson approximation.

Although the superposition model of central place hierarchy developed below includes the superposition, mixing and best fitting of the theoretical central place hierarchies, the underlying rationale is different – it based on the principle of superposition in the analysis of states of linear economic systems (Sonis, 1970, 1985, 1986; see, the chapter IV of this study): the superposition model of the of the central place hierarchy reflects the existence of different extreme tendencies of the spatial organization of central places, developing within an actual central place system. Thus, we will insert an actual central place hierarchy into the convex polyhedron of all admissible central place hierarchies. The vertices of this polyhedron are the extreme tendencies acting within an actual central place hierarchy. Each extreme tendency represent the mutual action of the optimal Christaller marketing, transportation and administrative principles, together with their Lössch generalizations in the form of Beckmann-McPherson Central place models. These models are the “building blocks” of the superposition model, which is a weighted sum (center of gravity = convex combination) of Beckmann-McPherson theoretical models. The weight of each Beckmann-McPherson Central place model represents a the degree of realization of the corresponding extreme tendency within the superposition of all relevant extreme tendencies. The competition and interference between different extreme tendencies generate the sequence of interdictions (“bottle neck” problems) generated by the collisions between the optimal Christaller-Lössch principles on the same hierarchical level of the actual central place hierarchy.

Below we will represent the complete theoretical treatment and the detailed computer algorithm for the construction of an actual central place hierarchy in

the form of the superposition model.

VI.2. The covering theorem.

The properties of hexagonal coverings of the plane in the Christaller-Loesch, Central Place theory are based on the following theorem from elementary geometry:

The covering theorem: *There are only three possible coverings of the plane by the regular polygons with n sides: by triangles ($n=3$), quadrates ($n=4$) and hexagons ($n=6$).*

VI.3. The construction of the central place geometry on a basis of barycentric coordinates on a plane.

The barycentric coordinates, i.e., coordinates of the center of gravity, are connected to the concept of the center of gravity introduced at first by Archimedes in the second century B.C. The barycentric coordinates appeared in the remarkable book of Möbius, 1837, as a basis for a projective geometry. The construction of the barycentric coordinates in a plane is based on a choice of the Möbius triangle within the Möbius plane. This plane is in the two-dimensional space defined by three barycentric coordinates, x, y, z , $x + y + z = 1$. The scale element of this plane is the Möbius equilateral triangle with the unit scale on each side.

VI.4. The Kanzig-Dacey formula.

If the point (v, u, w) is the origin $(0, 0, 1)$ of the lattice then the square of distance between (x, y, z) and $(0, 0, 1)$ gives the Kanzig-Dacey formula for the nesting factors in the Loeschian central place landscape:

$$k = x^2 + y^2 + xy$$

VII. The superposition model of central place hierarchy.

The superposition model of central place hierarchy is the application of the formalism of the Superposition Principle (see Ch. IV) to the analysis of the structure of an actual central place system. At first we describe the dual hierarchical structures of the central place system; then we immerse an actual central place system into the convex polyhedron of all admissible central place system. This immersion gives the possibility to apply the analytical formalism of the decomposition of an actual central place hierarchy into the convex combination of the Beckmann-McPherson extreme hierarchies which are the results of the Parr “best fitting” procedure. An important

example is the analysis of the original Christaller Munich Central place system. Furthermore, the polyhedral catastrophic dynamics of the central place hierarchies are described.

The hierarchy of hinterlands (market areas) is a “hierarchy by inclusion”, or by the size of market areas: the market areas of the same size belong to the same hierarchical level, and the order of hierarchical levels and the dominance relationships are defined by the inclusion of the market area of a smaller size in the market area of a bigger size. This hierarchy implies the triplicate interpretation of variable nesting factors: i) the nesting factor is the ratio of areas of hinterlands belonging to the different consecutive hierarchical levels; ii) the nesting factor is the number of market areas of the j th hierarchical level included in only one market area of $(j+1)$ th hierarchical level; iii) the nesting factor is the ratio of frequencies of market areas from j th and $(j+1)$ th hierarchical levels. The numerical description of the market place hierarchy can be given by the vector of market place frequencies in the actual central place system:

$m = (m_1, m_2, \dots, m_{n-1}, 1)$, where n is the number of hierarchical levels in a central place system and $m_j, j = 1, 2, \dots, n$, is the frequency of market areas from j th level. The ratios

$$k_j = \frac{m_j}{m_{j+1}}, j = 1, 2, \dots, n-1 \quad (\text{VII.1})$$

are the variable nesting factors. In the Christaller central place system

$k_1 = 3, 4, 7; k_2 = 9, 16, 49, \dots, k_m = 3^m, 4^m, 7^m$; in the Losch or in the

Beckmann-McPherson central place system k_j are the Kanzig-Dacey integers:

$$k_j = 1, 3, 4, 7, 9, 12, 13, 16, 19, \dots ;$$

in an actual central place system the nesting factors are arbitrary positive numbers, not necessary integers.

It is obvious that

$$m_j = k_j k_{j+1} \dots k_{n-1}, \quad j = 1, 2, \dots, n-1 \quad (\text{VII.2})$$

VII.2. The polyhedron of admissible central place hierarchies for an actual central

place system.

Let us consider an actual central place system given by a vector of market area frequencies $m_0 = (m_1^0, m_2^0, \dots, m_{n-1}^0, 1)$ or by the sequence

$$k_0 = (k_1^0, k_2^0, \dots, k_{n-1}^0) \quad (\text{VII.4})$$

of average nesting factors calculated with a help of the formula (VII.1). For the evaluation of the hierarchical structure of an actual central place system we shall put it into the convex polyhedron of all admissible central place hierarchies. For this, we will choose on each hierarchical level j the pair of Kanzig-Dacey theoretical nesting factors K_j, K'_j in such a way that the segment $[K_j, K'_j]$ will include the average nesting factors k_j^0 : $K_j \leq k_j^0 \leq K'_j$. This choice of theoretical nesting factors defines the convex polyhedron of all admissible central place hierarchies: it includes all sequences of average nesting factors $k = (k_1, k_2, \dots, k_{n-1})$ such that:

$$K_j \leq k_j \leq K'_j, j = 1, 2, \dots, n-1 \quad (\text{VII.5})$$

This system of inequalities presents geometrically the $(n-1)$ -dimensional rectangular parallelepiped, whose vertices have the integer Kanzig-Dacey coordinates K_j or K'_j ; thus, these vertices correspond to the Beckmann-McPherson central place models. The actual central place hierarchy (VII.4) corresponds to the inner point of this polyhedron. Let us introduce the slake variables, presenting the deflection of some central place hierarchy from the theoretical one on each hierarchical level j :

$$y_j = k_j - K_j \geq 0; \quad z_j = K'_j - k_j \geq 0, \quad j = 1, 2, \dots, n-1 \quad (\text{VII.6})$$

Then each admissible central place hierarchy $k = (k_1, k_2, \dots, k_{n-1})$ can be presented as a three-row matrix with non-negative components:

$$X = \begin{bmatrix} k_1 & k_2 & \dots & k_{n-1} \\ y_1 & y_2 & \dots & y_{n-1} \\ z_1 & z_2 & \dots & z_{n-1} \end{bmatrix} \quad (\text{VII.7})$$

and the actual central place hierarchy (VII.4) corresponds to the matrix

$$X_0 = \begin{bmatrix} k_1^0 & k_2^0 & \dots & k_{n-1}^0 \\ k_1^0 - K_1 & k_2^0 - K_2 & \dots & k_{n-1}^0 - K_{n-1} \\ K_1' - k_1^0 & K_2' - k_2^0 & \dots & K_{n-1}' - k_{n-1}^0 \end{bmatrix} \quad (\text{VII.8})$$

VII.3. The decomposition of an actual central place hierarchy.

According to the Superposition Principle (see Ch. IV), the hierarchical analysis of an actual central place system represented by the non-negative matrix X_0 is reduced to the decomposition of this matrix into the weighted sum of matrices X_1, X_2, \dots, X_{r+1} :

$$X_0 = p_1 X_1 + p_2 X_2 + \dots + p_{r+1} X_{r+1}, \quad r \leq n \quad (\text{VII.9})$$

where each matrix X_i represents the extreme state of the central place system, corresponding to some Beckmann-McPherson model and the weights p_i have a property:

$$p_1 + p_2 + \dots + p_{r+1} = 1; 0 \leq p_i \leq 1; r \leq n \quad (\text{VII.10})$$

If we take into consideration only the first row of each matrix in the decomposition (VII.9), we obtain the decomposition of the actual central place hierarchy

$k_0 = (k_1^0, k_2^0, \dots, k_{n-1}^0)$ into the convex combination of the Beckmann-McPherson central place hierarchies k_i with the same weights p_i :

$$k_0 = p_1 k_1 + p_2 k_2 + \dots + p_{r+1} k_{r+1}, \quad r \leq n \quad (\text{VII.11})$$

We interpret the decomposition (VII.9-11) in the following way: in each actual central place system there is a set of substantially significant tendencies towards the optimal organization of space in the form of Beckmann-McPherson hierarchies. Geometrically these tendencies define the simplex enclosed into the polyhedron of admissible central place hierarchies whose vertices correspond to the assemblage of the matrices X_i . An actual central place hierarchy X_0 is the center of gravity of this simplex with the

weights p_i . It is possible to interpret the weights p_i in a probabilistic form as the frequencies of the partial realization of some combination of the Chistaller-Loesch optimization principles in the hierarchical structure of the actual central place system.

The important fact is the non-uniqueness of the decomposition (VII.9-11) which follows from the existence of a set of different simplexes including the actual hierarchy X_0 . This non-uniqueness ensues from the fundamental methodological principle that the description of an actual state of a complex system under discussion depends on the point of view of investigator (Sonis, 1982; see also Ch.IV). Our view-point in this chapter the point of the best approximation of an actual central place hierarchy be the set of closest Beckman-McPherson models. Analytically this means that in the decomposition

(VII.9-11) the weight p_i will be the biggest possible and the following condition holds:

$$p_1 + p_2 + \dots + p_{r+1} = 1; 0 < p_{r+1} \leq \dots \leq p_2 \leq p_1 \leq 1; r \leq n \quad (\text{VII.12})$$

VII.4. The best fitting approximation procedure and the algorithm of decomposition.

The best fitting procedure of this chapter is a simplification of the procedure proposed by Parr, 1978a. This procedure will be used for the derivation of the central place hierarchy on each hierarchical level and in this way will be the basis for the construction of the best fitting simplex which contains the actual central place hierarchy matrix X_0 corresponding to the vector $k_0 = (k_1^0, k_2^0, \dots, k_{n-1}^0)$ of average nesting factors. The best fitting procedure is as follows: for each hierarchical level i the segment $K_i \leq k_i^0 \leq K_i'$ between the theoretical Kanzig-Dacey nesting factors K_i, K_i' can be chosen, which includes the average nesting factor k_i^0 . In this way the first best fitting Beckman-McPherson model $k_1 = (k_1^1, k_2^1, \dots, k_{n-1}^1)$ can be constructed with the help of "best fitting" formulae:

$$k_i^1 = \begin{cases} K_i & \text{if } k_i^0 \leq \frac{K_i + K_i'}{2} \\ K_i' & \text{if } k_i^0 > \frac{K_i + K_i'}{2} \end{cases} \quad (\text{VII.13})$$

In this procedure the values $\frac{K_i + K'_i}{2}$ define the boundaries of the domain of structural stability of the decomposition (VII.9-11).

The weight p_i of the Beckmann-McPherson model X_1 can be found by the requirement to choose the biggest positive p_i ($0 < p_i < 1$) satisfying the condition $X_0 - p_1 X_1 \geq 0$.

The place of the components of the matrices X_0 and X_1 , giving the minimum in (VII.14), defines the hierarchical level on which there exists the strongest interdiction to the extreme tendency represented by the chosen Beckmann-McPherson model X_1 , on part of other tendencies acting in the actual central place hierarchy.

The residual X' , defined by the equality

$$X_0 - p_1 X_1 = (1 - p_1) X' \quad (\text{VII.15})$$

represents the mutual action of other tendencies developing in the central place hierarchy with the weight $1 - p_1$. This means geometrically that we construct a straight line that passes the vertex X_1 and the point X_0 of the actual central place hierarchy and crosses the opposite face of the parallelepiped of admissible central place hierarchies at the point X' . Moreover, if one hangs the weights p_1 and $1 - p_1$ on points X_1 and X' then the center of gravity of the segment with end points X_1 and X' will coincide with the point X_0 .

For study of the residual X' , one should apply the previous "best fitting" procedure to the X' , and so forth..

VII.6. Hierarchical analysis of the Christaller original central place system in Munich, Southern Germany.

After the decades of empirical studies, the pure Christaller- Losch theoretical hierarchies of several hierarchical levels with the same nesting factors, have rarely if ever observed. The reason for this is that each actual central place hierarchy is the superposition of various theoretical hierarchies. It is interesting to see that even Christaller's original study of the Munich central place hierarchy confirms the phenomenon of superposition.

The Christaller original Munich central place hierarchy can be presented (see Woldenberg, 1979, Table V, p. 446) with the help of the following vector of market area

frequencies $m_0 = (519, 249, 127, 39, 12, 3, 1)$ with the corresponding sequence of average nesting factors $k_0 = (2.0843, 1.9606, 3.2564, 3.25, 4, 3)$. The polyhedron of admissible central place hierarchies is defined by the inequalities:

$$\begin{aligned} K_1 = K_2 = 1 &\leq k_1, k_2 \leq 3 = K'_1 = K'_2 \\ K_3 = K_4 = 4 &\leq k_3, k_4 \leq 4 = K'_3 = K'_4 \\ K_5 = K'_5 &= 4 = k_5 \\ K_6 = K'_6 &= 3 = k_6 \end{aligned}$$

This polyhedron includes all matrices of the form (see VII.8):

$$X = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 = 4 & k_6 = 3 \\ k_1 - 1 & k_2 - 1 & k_3 - 3 & k_4 - 3 & 0 & 0 \\ 3 - k_1 & 3 - k_2 & 4 - k_3 & 4 - k_4 & 0 & 0 \end{bmatrix}$$

The Munich central place hierarchy is represented by a matrix:

$$X_0 = \begin{bmatrix} 2.0843 & 1.9606 & 3.2564 & 3.25 & 4 & 3 \\ 1.0843 & 0.9606 & 0.2564 & 0.25 & 0 & 0 \\ 0.9157 & 1.0394 & 0.7436 & 0.75 & 0 & 0 \end{bmatrix}$$

The result of the analysis of the actual central hierarchy of Munich is:

$$\begin{aligned} X_0 &= 0.4803X_1 + 0.2633X_2 + 0.1946X_3 + 0.0554X_4 + 0.0064X_5 = \\ &= 0.4803 \begin{bmatrix} 3 & 3 & 3 & 3 & 4 & 3 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} + 0.2633 \begin{bmatrix} 1 & 1 & 3 & 3 & 4 & 3 \\ 0 & \bullet & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} + \\ &+ 0.1946 \begin{bmatrix} 1 & 1 & 4 & 4 & 4 & 3 \\ 0 & \bullet & 1 & 1 & 0 & 0 \\ 2 & 2 & \bullet & 0 & 0 & 0 \end{bmatrix} + 0.0554 \begin{bmatrix} 3 & 1 & 4 & 4 & 4 & 3 \\ 2 & \bullet & 1 & 1 & 0 & 0 \\ \bullet & 2 & \bullet & 0 & 0 & 0 \end{bmatrix} + \\ &+ 0.0064 \begin{bmatrix} 3 & 1 & 4 & 3 & 4 & 3 \\ 2 & \bullet & 1 & \bullet & 0 & 0 \\ \bullet & 2 & \bullet & 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

The first row of this matrix equality gives the decomposition of the vector of average

nesting factors:

$$\begin{aligned}
 k_0 &= (2.0843, 1.9606, 3.2564, 3.25, 4, 3) = \\
 &= 0.4803k_1 + 0.2633k_2 + 0.1946k_3 + 0.0554k_4 + 0.0064k_5 = \\
 &= 0.4803(3, 3, 3, 3, 4, 3) + \\
 &+ 0.2633(1, 1, 3, 3, 4, 3) + \\
 &+ 0.1946(1, 1, 4, 4, 4, 3) + \\
 &+ 0.0554(3, 1, 4, 4, 4, 3) + \\
 &+ 0.0064(3, 1, 4, 3, 4, 3)
 \end{aligned}$$

These decompositions means that the Munich central place hierarchy consists of five extreme tendencies. The first most prominent tendency corresponds to the Beckmann-McPherson model with nesting factors $k_1 = (3, 3, 3, 3, 4, 3)$. This tendency consists of the economizing of the number of market areas on almost each hierarchical level; only the second hierarchical level corresponds to economizing of transportation routes. This tendency is very closed to a perfect Christaller hierarchy $(3, 3, 3, 3, 3, 3)$ and maybe, this was a reason for the introduction by Christaller of his market principle. Nevertheless, the weight of this extreme tendency is equal to $p_1 = 0.4803$ only, i.e., it accounts only for 48.03% of the actual central place phenomenon. The second extreme tendency, corresponding to the Beckmann-McPherson model with the vector of nesting factors $k_2 = (1, 1, 3, 3, 4, 3)$, interdicts the first tendency on three lower hierarchical levels and represents the tendency of merging of these hierarchical levels, since the vector of nesting factors k_2 includes the nesting factors equal to 1. The second extreme tendency accounts for an additional 26.33% of the phenomenon. The third extreme tendency $k_3 = (1, 1, 4, 4, 4, 3)$ counteracts the first and second tendencies by implying the passage from market principle to the transportation principle on the fourth and fifth hierarchical levels. It explains additionally 19.46% of the phenomenon, so first three extreme tendency together explain 93.82% of the actual central place hierarchy. The fourth and fifth extreme tendencies are not so essential, since they explain together only 6.18% of the rest of phenomenon.

It is possible to present the cumulative action of the market and transportation optimization principles of all extreme tendencies separately on each hierarchical level, by accounting the weight of nesting factors 3 and 4 on each hierarchical level. In this

way we see that on the six hierarchical level only the market optimization principle is acting; on fifth level only the transportation principle appears; on the third and fourth hierarchical levels the market and transportation principles are acting in proportion 75%/25%. On the first and second hierarchical levels the market principle counteracted by the tendency of merging of these hierarchical levels.

Thus, the decomposition analysis of the Christaller example of the Munich, South Germany, central place hierarchy, , hints on the origins of appearance of Christaller optimization principles in the Central Place Theory.

VIII. Further directions of a study.

Here we describe some further directions of this study, which are not reflected in this synopsis. The form of the superposition principle is the Feedback Loops decomposition analysis of hierarchy of spatial production cycles in spatial economic system represented by matrices of economic flows (see Sonis and Hewings, 2001). (This hierarchy of feedbacks and satisfying the “matrioshka” imbedding principle representing the different levels of aggregation of flows). The hierarchy of spatial/functional linkages is visualized with the help of artificial economic landscapes based on minimum information multiplier product matrices (see Sonis and Hewings, 1997).

The ideas of Combinatorial Topology in the form of structural Q-analysis are used for the analysis of interconnections between the chains of interregional flows (see Sonis *et al*, 1999)

The complication of networks as synergetic augmentation process representing the dynamics of self-organization in multi-regional Input-Output systems is presented in Sonis and Hewings, 1998a,b.

REFERENCES:

Beckmann M J, 1958. “City hierarchies and the Distribution of city Size”, *Economic Development and Cultural change*, 6, pp. 243-248.

Beckmann M J and McPherson J C, 1970. ”City size distribution in the Central Place hierarchy: an alternative approach”, *Journal of Regional Science*, 10, pp. 243-248.

Boltiansky V G, 1973. *Optimal Control of Discrete Systems*, “Nauka”, Moscow.

Caratheodory C, 1911. “Uber den Variabilitats Bereich der Fourier’schen Constanten von positiven harmonischen Functionen, *Rend. Circ. Mat.* 32, Palermo, s. 198-201.

Christaller W, 1933. *Die Zentralen Orte in Suddeutschland*, Fischer, Jena; 1966, english

translation from German original by L W Baskin, *Central places in Southern Germany*, Prentice Hall, Englewood Cliffs, NJ.)

Christaller W, 1950. "Das Grundgerüst der räumlichen Ordnung in Europa", *Frankfurter Geographische Hefte*, 24, s. 1-96.

Cohon J L, 1978. *Multiobjective Programming and Planning*, Academic Press, New York.

Cowan G A, Pines D and Meltzer D, 1994. *Complexity, Metaphors, Models, and Reality*, Santa Fe Institute, Studies in the Science of Complexity, vol. XIX, Addison-Wesley, NY.

Dacey M F, 1964. "A note of Some Number Properties of a Hexagonal Hierarchical Plane lattice", *Journal of Regional Science*, 5, pp. 63-67.

Dacey M F, 1965. "The Geometry of Central Place Theory", *Geografiska Annaler*, 47, pp. 111-124.

Dacey M F, 1970. "Alternative Formulations of central Place Population", *Tijdschrift voore Economische en Socials Geografie*, 61, pp. 10-15.

Dantzig G B, 1951. "Application of the simplex method to a transportation problem." In T C Koopmans (ed), *Cowles Commission Monograph 13: Activity Analysis of Production and Allocation*. John Wiley, NY, pp. 359-373.

Dantzig G B, 1963. *Linear Programming and Extensions*. Princeton University Press, Princeton, NY.

Kantorovitch L V, 1942. "On dislocation of masses." *Russian Doklady of the USSR Academy of Sciences*, 37, no.3., pp. 227-229.

Kantorovitch L V, Gavurin M K, 1949. "Application of mathematical methods to the analysis of commodity flows". In *Problems in the rise of effectiveness of Transport Activity*, USSR, Moscow, Academy of Sciences, pp. 110-138, in Russian.

Lee E S, 1966. "A Theory of Migration". *Demography*, 3, pp.47-57.

Lösch A, 1940. *Die Räumliche Ordnung der Wirtschaft*, Fischer, Jena; 1944, second edition; 1962, third edition, Fisher, Stuttgart; 1954, english translation from German original by W H Woglom and W F Stolper, *The Economics of Location*, Yale University Press, New Haven, Connecticut.; second print, Wiley, New York.

Minkovski H, 1910. *Geometrie der Zahlen*, B.G. Tenbner, Leipzig, Berlin (English Translation, 1953, *Theory of Numbers*, Chelsea Publ, New York).

Marshall J U, 1977. "The Construction of L&schian Landscape", *Geographical Analysis*, 9, pp.1-13.

Moebius A F, 1827. *Der Barycentrische Calcul*, Leipzig.

Nystuen J D and M F Dacey, 1961. "A Graph-Theory Interpretation of Nodal Regions." *Papers and Proceedings of the Regional Science Association*, 7, pp. 29-42.

Parr J B, 1970. "Models of City Size in an Urban System", *Papers of Regional Science Association*, 25, pp. 221-253.

Parr J B, 1978a. "An alternative Model of the Central Place System". In Batey P W J (ed) *London papers in Regional Science*, 8. *Theory and Method in Urban and Regional Analysis*. Pion, London, pp. 31-45.

Parr J B, 1978b. "Models of Central Place System: A More General Approach, *Urban Studies*, 15, pp. 35-49.

Parr J B, 1981. "Temporal Change in a Central place System, *Environment and planning A*, 13, pp. 97-118.

Parr J B, Denike KG and G Milligan, 1975. "City Size Models and the economic basis: A Recent Controversy", *Journal of Regional Science*, 15, pp. 1-8.

Ravenstein E G, 1885. "The Laws of Migration." *Journal of the Royal Statistical Society*, 48, pp. 167-227.

Sonis M, 1971. "Sensitivity analysis and Multiparametric Programming". Tel Aviv University, Department of Pure Mathematics, October 1971, unpublished manuscript.

Sonis M, 1980. "Locational Push-Pull Analysis of Migration Streams", *Geographical Analysis*, 12, 1, pp. 80-97.

Sonis M, 1982a. "Domains of Structural Stability for minimal cost Discrete Flows with reference to Hierarchical Central-Place Models." *Environment and Planning A*, 14, pp. 455-469

Sonis M, 1982b. "The Decomposition Principle versus Optimization in Regional Analysis: The Inverted Problem of Multiobjective Programming". In G Chiotis, D Tsoukalas and H Louri (eds), *The Regions and the Enlargement of the European Economic Community*, Athens, Eptalofos, pp. 35-60.

Sonis M, 1985. "Hierarchical structure of Central Place System – the barycentric calculus and decomposition principle." *Sistemi Urbani*, 1, pp. 3-28

Sonis M, 1986a. "A contribution to the Central Place Theory: superimposed Hierarchies, Structural Stability, Structural changes and Catastrophes in Central place Hierarchical Dynamics." In R. Funk, A. Kuklinsky (eds) *Space-Structure -Economy: A Tribute To August Leosch*, Karlsruhe Papers in Economic Policy Research, 3, 159-176.

Sonis M, 1986b. "Transportation Flows within Central-Place Systems." In D.A. Griffith, R.A. Haining (eds). *Transformations through Space and Time*, Martinus Nijhoff, Amsterdam, 81-103.

Sonis M, 1993. "Optimal extensions of the transportation flows and competition between suppliers and demanders." *Sistemi Urbani* 1, pp. 3-15.

Sonis M, G.J.D. Hewings. 1998. "Economic Complexity as Network Complication: Multiregional Input-Output Structural Path Analysis", **The Annals of Regional Science**, **32:407-436**.

Sonis M, G.J.D. Hewings, 1998. "The Temporal Leontief Inverse". **Macroeconomic Dynamics**, 2, 89-114.

Sonis M, G.J.D. Hewings, 1999. "Economic Landscapes: Multiplier Product Matrix Analysis for Multiregional Input-Output Systems", **Hitotsubashi Journal of Economics** vol.40, no.1, pp.59-74.

Sonis M, G.J.D. Hewings and J-M. Guo, 2000. "A new image of classical Key Sector analysis: Minimum Information decomposition of the Leontief Inverse", **Economic Systems Analysis**, vol.12, no.3, pp.401-423.

Sonis M, 2000. "Catastrophe effects and optimal extensions of Transportation flows in the developing Urban System: A Review". In D. Helbing, H.J.Herrmann, M. Schreckenberg, D. E. Wolf (eds) **Traffic and Granular Flows '99**, Social, Traffic and Granular Dynamics, Springer, pp.31-41.

Sonis M, G.J.D. Hewings and Y. Okuyama, 2000, "Vertical specialization and interregional trade: Spatial Production Cycles and Feedback Loop Analysis of the Midwest Economy." In H. Herrmann and J. Broecker (eds) **Regional Research in an International Perspective**, Springer Verlag.

Sonis M, G.J.D. Hewings, 2001. "Feedbacks in Input-Output System: impacts, loops and hierarchies." Chapter 4 in M. Lahr and E. Dietzenbacher (eds) **Input-Output Analysis: Frontiers and Extensions**, Palgrave, pp. 71-99..

Sonis M, 2000. "Non-Linear Socio-Ecological Dynamics and First Principles of

Collective Choice Behavior of “Homo Socialis”, **Progress of Theoretical Physics Supplement**, No. 139, pp.257-269.

Sonis M, 2001. “Major Actors in Innovation Diffusion Process”. Chapter 16 in M. Fischer and J. Froelich (eds) **Complexity, Knowledge and Innovation Systems** , Springer. Pp. 317-341.

Stouffer S A, 1960. “Intervening Opprtunities and Competing Migrants.” *Journal of Regional Science*, 2, pp. 1-26.

Tinkler K, 1978. “A Co-ordinate System for Studying Interactions in the Primary Christaller Lattice”, *Professional Geographer*, 30, pp. 135-139.

Weber A, 1909. *Uber den Standort der Industrien*, Tubingen.

Weyl H, 1935. “Elementare Theorie der konvexen Polyeder.” *Commentarii Mathematici Helvetici* 7, pp. 290-366, (English translation: *Contributions to the Theory of Games*, 1 (1950), Princeton, pp. 3-18).

Woldenberg M J, 1968. “Energy flows and Spatial Order – Mixed Hexagonal Hierarchies of Central Places”, *Geographical Review*, 58, pp. 552-574.

Woldenberg M J, 1979. “A Periodic Table of Spatial Hierarchies”. In Gale S and

Ollson G (eds) *Philosophy in Geography*, D Reidel Publ. Company, Dordrecht, Holland, pp. 429-456.