Central Place Theory after Christaller and Lösch: Some further explorations.

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In memory of Alfred Lösch, 15 October 1906- 30 May 1945.

ABSTRACT. This paper deals with the critical reevaluation of the methodology of classical Christaller - *Lösch* Central Place Theory**.** In the beginning of the paper the reconstruction of Central Place Geometry on the basis of the *Möbius* Barycentric Calculus was considered. On this basis a superposition model of the actual Central Place System is constructed. Building blocks of this model are the Beckman-McPherson models representing the main tendencies of optimal organizations of space acting simultaneously in the actual Central Place system. The weights of these building blocks represent the level of realization of the specific extreme tendencies in the real system. The algorithm of decomposition of an actual Central place system into the weighted sum of the Beckman-McPherson building blocks is elaborated and presented in detail. This algorithm generates also the description of the interconnections of hexagon coverings on the sequential hierarchical levels with the help of convex combinations of hexagon coverings and homothetic transformation of the coverings.

Next, the (jumping) catastrophic dynamics of the Central Place hierarchies presented with the help of geometrical scheme of the movement of the point representing actual Central Place system in the polyhedron of all admissible Central Place systems.

Two main applications of this conceptual framework are elaborated:

- The enumeration of all structurally stable optimal (minimal cost) transportation flows in the hierarchical Central Place system and
- The merger of two major theories in the Regional Science: the classical Input-Output theory of Leontief and the classical Christaller - *Lösch* Central Place theory.

We hope that this critical reevaluation of the geometrical and conceptual basis of Central Place theory will contribute to narrowing the existing gap between the formal theory and empirical studies.

Key words: Central Place theory, Barycentric Calculus; Superposition Model of Central place Hierarchy; Jumping Catastrophe Dynamics of Central Place Hierarchy; Structural Stability of Transportation flows in the Central Place system**;** The merger of Input-Output theory and Central Place theory.

1. Barycentric Calculus and Superposition Model of Central Place Hierarchy

The Central Place theory established itself as one of the most influential theories of theoretical geography and theoretical spatial economic analysis. The concepts and methodological basis of Central Place theory were formulated in the first part of previous century by two scientists in Germany: geographer Walter Christaller (1933) and economist August Lösch (1940).

The ideas of Christaller were first introduced into the English language by Ullman (1941). In 1954 appeared the English translation of the book of Lösch and in 1966 the translation of the book of Christaller. Since then, the concept of central place hierarchy captivated the imagination of spatial analysts. Empirical evaluation of the ideas of the Central Place theory began with papers by Brush and Bracey, 1955, and by Berry and Garisson, 1958, which have influenced many later empirical studies. It is possible to find a review of the early work in the Central Place theory and its applications in studies of Berry and Garrison, 1958, Berry and Pred, 1961 and in the books by Bunge, 1962, Lloyd and Dicken, 1977 and Beavon, 1977.

It is important to note that from the first steps of the Central place theory a gap emerged between the formal theory and empirical studies. The need to close this gap caused the appearance of critical methodological studies of the logic of the Central Place theory in the form of axiomatic method. The leading role in the developing of the formal axiomatic approach to the construction of the theory of Central Places belongs to the American geographer Michael Dacey who in 1960ies-1970ies initiated (Dacey, 1964, 1965, 1970) and inspired the studies of a large group of geographers (Dacey and Sen, 1968, Dacey, Davies, Flowerdew, Huff, Ko, Pipkin, 1974; Alao, Dacey, Davies, Denike, Huff, Parr, Webber, 1977). Despite of the initial enthusiasm and big promises their work was heavily based on the geometrical ideas of two geometrical texts by Hilbert and Con-Fossen, 1932 (English translation 1952) and Coxeter, 1961, (both out of date now). The axiomatic approach became formal and abstract and did not influence the new empirical studies of actual Central Place systems. The gap between the theory and empirical studies remains open till now. Although at present there is no doubt about the conceptual usefulness of the Central Place theory, its essential deficiency relates to its applicability to the analysis of an actual central place system. Moreover, the classical Central Place theory represents the challenge to the New Urban Economics and New Economic Geography which both fail to reproduce and incorporate the spatial basis of the classical Central Place theory (*cf.* David, 1999, Fujita, Krugman and Venables, 1999).. In this paper we try to close the existing gap between the pure theoretical Christaller and Lösch models and the empirical structure of an actual central place system; we present an alternative hierarchical model based on the idea of mixed hierarchy of the Central Place system (Christaller, 1950, p.12; Woldenberg, 1968) and on the Beckmann-McPherson model of Central Place system (Beckmann, McPherson, 1970), which are the intermediate links between the Christaller and Lösch models.

1. Elements of the Central Place geometry

The spatial description of the original Chrisraller Central Place model is based on following generic geometric properties of central places associated with Central Place system:

1. *The first property is that all hinterland areas of the central places at the same hierarchical level form a hexagonal covering of the plane with the centers on the initial homogeneous triangular lattice presenting the centers of the hexagons from the Christaller primary covering.* The properties of hexagonal coverings of the plane in the Christaller - Lösch Central Place theory are based on the following theorem from elementary geometry:

The covering theorem*: There are only three possible coverings of the plane by the regular polygons with* n *sides: by triangles (n=3), quadrates (n=4) and hexagons (n=6).*

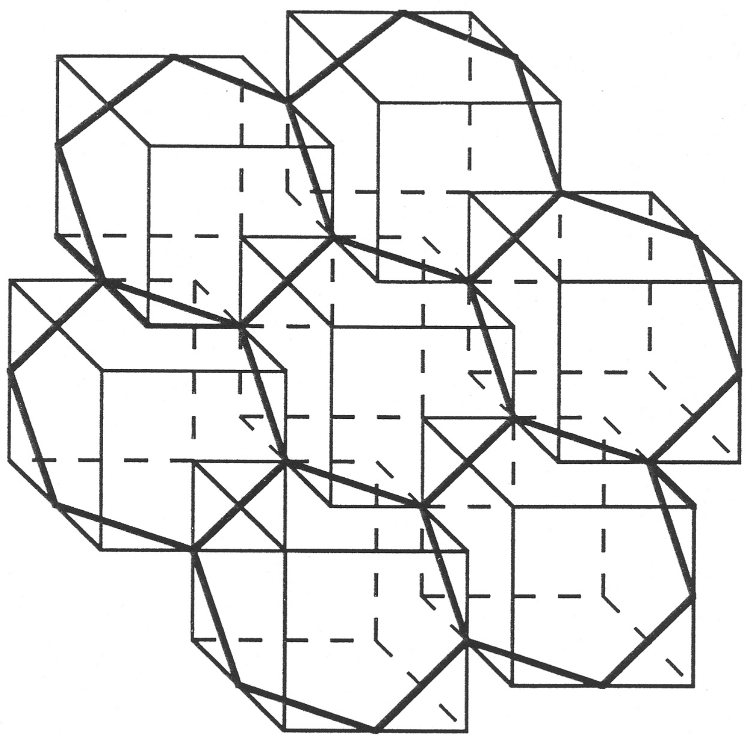


Figure 1. Derivation of the hexagonal covering of the plane by section of the arrangement of a layer of cubes in space

The covering theorem was known to Pythagoreans in V Century B.C. Figure 1 demonstrated the interconnection between the filling of space by a layer of cubes and the hexagon covering of the plane: the section of the three-dimension arrangement of layer of cubes by the plane gives the covering of the plane by regular hexagons. This three-dimensional arrangement of a layer of cubes includes cubes whose vertices are the centers of quadrate faces of adjacent cubes. This property of section of the arrangement of cubes will be used in the next chapter for the construction of interconnection of the system of barycentric coordinates in the Möbius plane and usual Euclidean metrics in space.

2. *The second property is that the size of the hinterland areas increases from the smallest (on the lower tier of Central Place hierarchy) to the largest (on the highest tier of hierarchy) by a constant nesting factor k.*

By definition, the nesting factor is the ratio between the area *S* of the hexagon belonging to some hexagonal covering of the plane to the area *s* of hexagon belonging to the primary Christaller covering by smallest hexagons with the property: the distance between the centers of smallest hexagons equals 1: $k = S / s$

It is easy to see that if *d* is the distance between the centers of adjacent hexagons of some hexagonal covering of the plane then the area of each hexagon is equal to $S = 2\sqrt{3}d^2$, so the area of smallest hexagon from the Christaller primary covering is equal to $s = 2\sqrt{3}$. Thus, the nesting factor equals to the square of the distance between the centers of adjacent hexagons of hexagonal covering of the plane: $k = d^2$.

3. *The third property is that the center of a hinterland area of a given size is also the center of hinterlands of each smaller size (Christaller, 1933). The nesting factors 3, 4, 7 play the most important role in the Christaller Central Place theory: they express one of the Christaller three principles, namely, marketing (k = 3), transportation (k = 4) and administrative (k = 7) principles.* The nesting factors 3, 4, 7 generate three geometrical sequences of the hexagonal market area sizes: 1, 3, 9, 27,…, 3^n,…; 1, 4, 16, 64,…, 4^n,…; 1, 7, 49, 343,…, 7^n. It is possible to interpret these Christaller principles as principles of optimal organization of the central place market areas: marketing principle represents the minimal number of small market areas – three **-** included in a bigger market area; the transportation principle presents such optimal organization of space where the transportation network between two bigger central places passes through the smaller central place; the administrative principle presents such optimal organization of space where the administrative hinterland of the larger central place includes almost completely the set of administrative hinterlands of smaller central places.

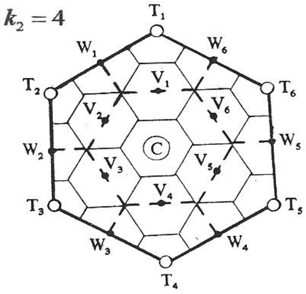
5. *The Löschian hexagonal landscape (Lösch,* 1940) *is the superposition of all possible coverings of a plane by hexagons whose centers are coincide with the vertices of the triangular lattice and the sizes of market areas (nesting factors) are integers:* k = 1, 3, 4, 7, 9, 12, 13, 16, 19,... The geometric procedure for construction of the Löschian landscape is simple and straightforward: for the derivation of a part of the Löschian landscape which corresponds to the hexagonal covering with a nesting factor $k = d^2$, one should chose on the Christaller primary lattice two points with the distance *d* between them, to derive the segment connected these two centers and from its middle point to draw a perpendicular segment of the size $d / \sqrt{3}$. The end point of this perpendicular segment is the vertex of the hexagon and, thus defines the position of whole hexagon and all hexagons from the corresponding coverings. Each hierarchical level in the Löschian landscape includes the primary hexagonal covering with its own geometric scale and secondary hexagon covering with a definite nesting factor built up on the primary covering. Lösch himself constructed the coverings corresponding to 150 nesting factors. By rotating of the different coverings Lösch show that in vicinity of an origin the market areas are arranged into six "activity (center) rich" and six "activity (center) poor" sectors. As Lloyd and Dicken, 1972, pp. 48-49, commented, "this *particular section of Lösch's work has been the subject of much controversy and misinterpretation…The work by Tarrant, 1973, and Beavon and Mabin, 1975, suggests a rather different interpretation…According to both studies, the production of "city-rich "and "city-poor" sectors is not the result of rotation, as many have believed, but a constant upon it. In other words, if the sectoral pattern is to be achieved there is a very limited number of ways in which the hexagonal net can be arranged. Once certain ones are oriented in a particular way the positions of the others are fixed."*

Moreover, as demonstrated by Marshall, 1977, this arrangement of "city-rich "and "city-poor" sectors is local and do not hold for the big distances from the origin. Parr indicated (Parr, 1970, p.45) that these Löschian landscape nesting factors also present the optimal organizations of space similar to Christaller marketing, transportation and administrative principle; for example, the nesting factors 13 and 19 have the same property of administrative convenience as factor 7, while factors 9 and 16 have the same transportation efficiency as factor 4. According to Lloyd and Dicken, 1972, p. 49, *" Lösch suggested that this spatial arrangement of urban centers was consistent with what he saw to be a basic element in human organization: the principle of least effort."*

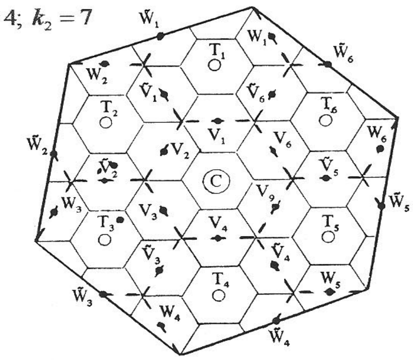
6. *The Beckmann-McPherson, 1970, Central Place model differs from the Christaller framework by applying variable nesting factors and by using the principle of possible*

coverings of the plane by hexagons of variable integer sizes. Their centers are the vertices of the initial Christaller triangular lattice.

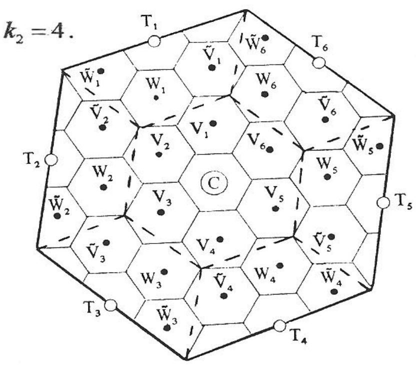
The Christaller model is only a partial case of Beckmann-McPherson models. Simultaneously, the Beckmann-McPherson models are an incomplete case of the *a).* $k_1 = 3;\ k_2 = 4$



(b). $k_1 = 4;\ k_2 = 7$



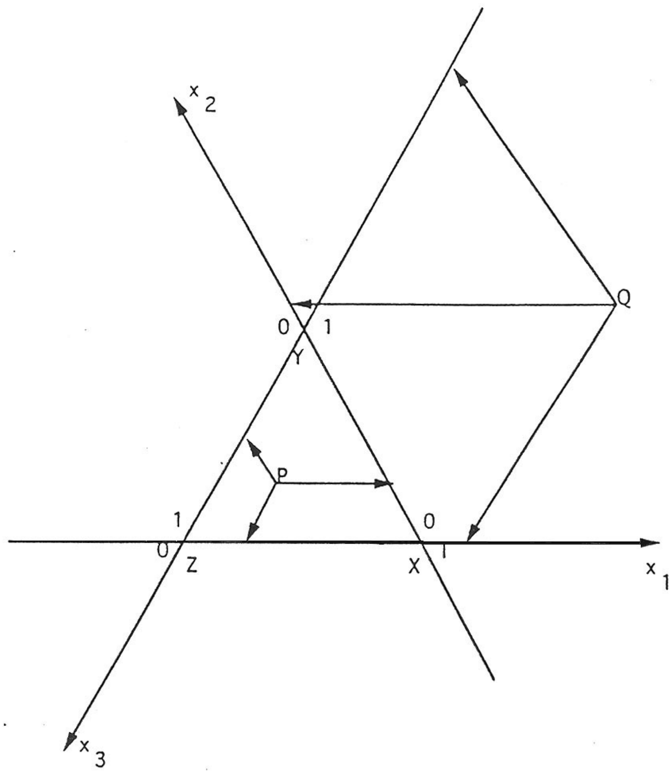
(c). $k_1 = 7;\ k_2 = 4$.



Löschian model – incomplete in the sense that the Beckmann-McPherson models include only a small part of the hinterland areas from the Löschian landscape (see figure 3). Parr, 1970, described the way to compare the theoretical models with the structure of the actual central place system. His idea was to use the Beckmann-McPherson Central place model as the best fitting approximation of an actual central place hierarchy. Parr also met with difficulties that arise from the omission of the analysis of the discrepancy between the actual central place hierarchy and its best fitting Beckmann-McPherson approximation.

3. The construction of the Central Place geometry on a basis of barycentric coordinates on a plane.

The barycentric coordinates, i.e., coordinates of the center of gravity, are connected to the concept of the center of gravity introduced at first by Archimedes in the second century B.C. The barycentric coordinates



appeared in the remarkable book by Möbius , 1837, as a basis for a projective geometry. The construction of the barycentric coordinates in a plane is based on a choice of the Möbius triangle within the Möbius plane. This plane is in the two-dimensional space defined by three barycentric coordinates $x, y, z, \quad x + y + z = 1$. The scale element of this plane is the Möbius equilateral triangle with the unit scale on each side. This triangle is generated by three coordinate axes (see figure 4).

Each covering of the plane by equal hexagons generates the system of barycentric coordinates corresponding to the Möbius triangle with different scales. It is possible to measure the barycentric coordinates of each point in the Möbius plane by projecting it (parallel to the sides) onto the sides of the Möbius triangle. If the point, *P*, lies within the Möbius triangle, then its barycentric coordinates, x, y, z must be between 0 and 1. The vertices of the Möbius triangle are:

$$X \; : x = 1, \; y = 0, \; z = 0; \quad Y \; : \; x = 0, \; y = 1, \; z = 0; \quad Z \; : \; x = 0, \; y = 0, \; z = 1.$$

The mechanical interpretation of the barycentric coordinates as coordinates of center of gravity (barycenter) is as follows: the point *P* with coordinates x, y, z is the center of gravity of the weights x, y, z hanging in the vertices X, Y, Z of the Möbius triangle.

If point *P* lies outside of the Möbius triangle. triangle (see figure 4) then one or two barycentric coordinates must be negative, but the condition $x + y + z = 1$ always holds.

The barycentric coordinates of the central places of the initial Christaller hexagonal coverings of the Möbius plane are positive or negative integers.

It is interesting to note that the barycentric coordinates appeared in a latent and mysterious form in the geometry of the Central Place theory – in the

form of the rhombic coordinates *x* and *y* in the primary Christaller triangular lattice (Dacey, 1964, 1965) or in the form of the coordinate triples (*x, y, x*+*y*), where *x, y* are the rhombic coordinates (Tinkler, 1978). Neither Dacey nor Tinkler realized that the triple (*x, y, z*) where $z = 1 - x - y$ present three barycentric coordinates in a Möbius plane.

3. The Kanzig - Dacey formulae.

The figure 1 points out on the possibility to present the barycentric coordinates on a Möbius plane as usual Euclidian coordinates on a plane $x + y + z = 1$ in three dimensional space. The equation $x + y + z = 1$ represents a plane in three-dimensional space based on the triangle with the vertices (5) which is the Möbius triangle (see figure 5). The transfer of the barycentric coordinates from plane to space increases the scale by the factor $\sqrt{2}$, and gives the simple way to obtain the Dacey formula for theoretical nesting factors (Dacey, 1964, 1965) $k = x^2 + y^2 + xy$ where $x, y, z = 1 - x$ are the barycentric coordinates of the central place: x, y are arbitrary positive and negative integers. To prove this formula we note that for different points (*x, y, z*) and (*v, u, w*) on the plane $x + y + z = 1$ the usual Euclidean distance *d* is:

$$Dist = \sqrt{(v-x)^2 + (u-y)^2 + (w-z)^2} = \sqrt{2[(v-x)^2 + (u-y)^2 + (v-x)(u-y)]}$$

The distance *d* between the central places (*x, y, z*) and (*v, u, w*) on the Möbius plane can be obtain from *Dist* by scaling in on parameter $\sqrt{2}$, i.e.

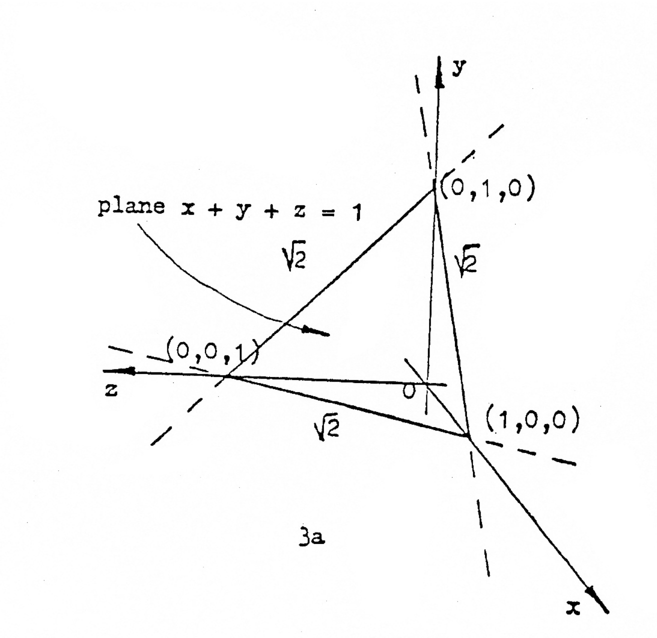
$$d = \sqrt{(v-x)^2 + (u-y)^2 + (v-x)(u-y)}$$

If the point (*v, u, w*) is the origin (0, 0, 1) of the lattice then the square of distance between (*x, y, z*) and (0, 0, 1) gives the Dacey generating formula for the nesting factors in the Löschian central place landscape:

$$k = x^2 + y^2 + xy \qquad (1)$$

Figure 5. The interconnection between the barycentric coordinates in the Möbius plane and the Euclidian coordinates in space.

where x and y are arbitrary integer numbers.



If we introduce the new parameters $u=x/2$ and $v=x/2 + y$ then the Dacey formula (1) will be equivalent to the Kanzig formula:

$$k = 3u^2 + v^2 \qquad (2)$$

where u and v are arbitrary half-integer numbers. Werner Kanzig presented his formula empirically in the English translation of the Lösch book "*The Economics of Location*", 1954, p.119. Beavon and Mabin, 1975, proved a correct form of the Kanzig formula. Both formulas of Dacey and Kanzig are generating the same sequence of the theoretical Lösch nesting factors: $k = 1, 3, 4, 7, 9, 12, 13, 16, 19, \ldots$

4. Barycentric calculus of the Löschian hexagonal landscape.

The universal geometrical procedure of the construction of all hexagonal coverings from Löschian hexagonal landscape (see chapter 1) can be presented with the help of barycentric coordinates of centers of hexagons: consider the center of the hexagon with integer coordinates (x, y, z), $x + y + z = 1$; construct the segment S connecting the point (x, y, z) with the point (0, 0, 1). The square d^2 of the distance d between these two points according to Kanzig-Dacey formula coincides with the nesting factor

$d^2 = k = x^2 + y^2 + xy$. Further, let us draw from the middle point of the segment S a perpendicular segment of the size $d/\sqrt{3}$. The end points of this perpendicular segment are the vertices of the hexagon and, thus define the position of whole hexagon covering of the plane corresponding to the nesting factor *k*.

Next we introduce two important operations with hexagon coverings:

4.1. Convex combination of hexagon coverings

Consider two different hexagon coverings based on the same Christaller primary covering. These two coverings can be constructed with the help of two points in the Möbius plane: $(x_1, y_1, z_1), (x_2, y_2, z_2)$. For arbitrary number (weight) α the convex combination of these points can be derived as a point with following barycentric coordinates $(\alpha x_1 + (1-\alpha) x_2, \alpha y_1 + (1-\alpha) y_2, \alpha z_1 + (1-\alpha) z_2)$

This point can be used for the construction of a new hexagon covering which will be called the convex combination of two hexagon coverings. In a similar manner can be constructed the convex combination of arbitrary number r of hexagon coverings with weights $\alpha_1, \alpha_2, ..., \alpha_r$, $\alpha_1 + \alpha_2 + ... + \alpha_r = 1$.

4.2. Homothetic transformation of hexagon coverings.

Consider the arbitrary hexagon covering, constructed with help of the point (x, y, z) in the Möbius plane, corresponding to nesting factor $k = x^2 + y^2 + xy$ and the positive number $r > 0$. The hexagon covering, constructed with the help of point $(\sqrt{r}x, \sqrt{r}y, 1-\sqrt{r}+\sqrt{r}z)$, is called a homothetic transformation of hexagon covering with radius of homothety *r.* The nesting factor k_r of the homothetic transformation of hexagon covering equals: $k_r = rk$

As will be shown further, the convex combinations of hexagon coverings and their homothetic transformations describe the transfer from one hierarchical level of Central place hierarchy to the next hierarchical level.

5. Dual hierarchical structures of the central place system.

Each central place system characterized by two dual hierarchical structures: a hierarchy of market areas (hinterlands) and a hierarchy of centers (central places) of market areas. The first hierarchy was used as a city-size model by Beckmann, 1958, whereas Dacey, 1970, has treated the second one without paying attention to dual interconnections between them. The duality of the two hierarchies was discovered by Parr in the form of a similarity between the Beckmann and Dacey city-size models (Parr, 1970; Parr, Denike, Mulligan, 1975).

The hierarchy of hinterlands (market areas) is a "hierarchy by inclusion", or by the size of market areas: the market areas of the same size belong to the same hierarchical level, and the order of hierarchical levels and the dominance relationships are defined by the inclusion of the market area of a smaller size in the market area of a bigger size. This hierarchy implies the triplicate interpretation of variable nesting factors:

- the nesting factor is the ratio of areas of hinterlands belonging to the different consecutive hierarchical levels;
- the nesting factor is the number of market areas of the *j*th hierarchical level included in only one market area of (*j*+1)th hierarchical level;
- the nesting factor is the ratio of frequencies of market areas from *j*th and (*j*+1)th hierarchical levels.

The numerical description of the market place hierarchy can be given by the vector of market place frequencies in the actual central place system:

$$\boldsymbol{m} = (m_1, m_2, ..., m_{n-1}, 1), \qquad (3)$$

where n is the number of hierarchical levels in a central place system and m_j, $j = 1, 2, ..., n$, is the frequency of market areas from *j*th level.

The ratios

$$k_j = \frac{m_j}{m_{j+1}},\ j = 1, 2, ..., n-1 \qquad (4)$$

are the variable nesting factors. It is obvious that

$$m_j = k_j k_{j+1} ... k_{n-1}, \quad j = 1, 2, ..., n-1 \qquad (5)$$

In the Christaller central place system

$$k_1 = 3,\ 4,\ 7;\quad k_2 = 9,\ 16,\ 49, ..., k_m = 3^m,\ 4^m,\ 7^m \qquad (6)$$

In the Lösch or in the Beckmann-McPherson central place system k_j are the Kanzig-Dacey integers: k_j = 1, 3, 4, 7, 9, 12, 13, 16, 19,... .The above-described hierarchy of market areas generates the dual hierarchy of the centers of market areas on the basis of duality correspondence: *Market area (hinterland of the central place)* ⇔ *Central place (center of market area)* such that the order *j* of the hierarchical level of a given central place is equal to the order of hierarchical level of the biggest market area with the same center; the dominance relationship between the centers is defined by the geometric inclusion of corresponding hinterlands. It is possible to give the analytical description of the hierarchy of centers of market areas by means of a vector of center frequencies

$$\boldsymbol{c} = (c_1, c_2, ..., c_{n-1}, 1) \qquad (7)$$

where c_j is the frequency of center from *j*th hierarchical level. The duality correspondence implies the connections between the vectors ***m*** of market area frequencies and vectors ***c*** of center frequencies:

$$\begin{aligned} c_j &= m_j - m_{j+1} = m_{j+1}\left(k_j - 1\right) = \left(k_j - 1\right)k_{j+1}...k_{n-1} \\ m_j &= c_j + c_{j+1} + ... + c_{n-1} + 1 \end{aligned} \qquad (8)$$

6. Empirical Average Central Place hierarchies.

In empirical studies of concrete Central Place systems the main measurable statistical data is the vector $c_0 = (c_1^0, c_2^0, ..., c_{n-1}^0, 1)$ of empirical center frequencies is main measurable statistical data. Formulae (11) and (7) give the coordinates of the vector of empirical market areas frequencies $m_0 = (m_1^0, m_2^0, ..., m_{n-1}^0, 1)$ and the coordinates of the vector of average nesting factors $k_0 = (k_1^0, k_2^0, ..., k_{n-1}^0)$ **.** The average nesting factors are the arbitrary positive numbers, not necessary integers.

Christaller, 1950, himself came to realize that the marketing, transportation and administration principles could be expected to act simultaneously in geographical space. He suggested modifying his original model by *a mixing* of the nesting factors 3, 4, and 7 into the grouping non-integer nesting factor k = 3.3 which generates the geometric progression 1, 3.3, 10, 33,. Woldenberg, 1968, elaborated on analogy between the hierarchical structure of fluvial systems and the hierarchical structure of the hinterlands of the central place systems, so as to be able to generate the sequences of average non-integer nesting factors for sizes of market areas for central place systems. With the help of numerical computer model Woldenberg, 1979, compared the results of computer simulations with a wide set of actual empirical central place hierarchies and mentioned certain difficulties that rise in attempting to describe an actual hierarchy in terms of the numerical computer model. The week points of these generic models are the non-uniqueness of the procedure of grouping and empirism in the underlying theoretical reasoning.

The empirical central place hierarchies generate in the vicinity of each central place the local nested geometric structure of average market areas, i. e. set of hexagons with the centers located in the given central place. The areas of these hexagons correspond to the vector of empirical market areas frequencies $m_0 = (m_1^0, m_2^0, ..., m_{n-1}^0, 1)$ generating the coordinates of the vector of average nesting factors $k_0 = (k_1^0, k_2^0, ..., k_{n-1}^0)$ The construction the geometrical base of this local hierarchy of empirical average market areas needs the elaboration of the theory of the superposition, mixing and best fitting of the theoretical central place hierarchies and the construction of the new superposition model of the of the central place hierarchy which reflects the existence of different extreme tendencies of the spatial organization of central places, developing within an actual central place system (Sonis, 1970, 1982, 1985, 1986). Therefore the geometry of local hierarchy of empirical average market areas will be presented in detail in chapter 9 after the introduction of the superposition model of the of the central place hierarchy.

7. The superposition model of central place hierarchy.

Now we will present a general superposition Central Place model with arbitrary number of hierarchical levels. For the construction of such generalization we will use the theory of convex polyhedra in multi-dimensional space (see Weyl, 1935)

The superposition model of central place hierarchy is the application of the formalism of the Superposition Principle (see Sonis, 1970, 1982b) to the analysis of the structure of an actual central place system. At first we immerse an actual average central place system into the convex polyhedron of all admissible central place system. This immersion gives the possibility to apply the analytical formalism of the decomposition of an average central place hierarchy into the convex combination of the Beckmann-McPherson extreme hierarchies (Beckmann-McPherson, 1970), which are the results of the Parr "best fitting" procedure (Parr, 1978a).

7.1.Polyhedron of Admissible Central Place Hierarchies for an Actual Central Place System

Let us consider an actual central place system given by a vector of market area frequencies $m_0 = (m_1^0, m_2^0, ..., m_{n-1}^0, 1)$ or by the sequence:

$$k_0 = (k_1^0, k_2^0, ..., k_{n-1}^0) \quad (9)$$

of average nesting factors calculated with a help of the formula (7). For the evaluation of the hierarchical structure of an actual central place system, we shall place it into the convex polyhedron of all admissible central place hierarchies. For this, we will choose on each hierarchical level, *j*, the pair of theoretical nesting factors $K_j, K_j^{'}$ in such a way that the segment [$K_j, K_j^{'}$] will include the average nesting factors k_j^0: $K_j \le k_j^0 \le K_j^{'}$. This choice of theoretical nesting factors defines the convex polyhedron of all admissible central place hierarchies: it includes all sequences of average nesting factors $k = (k_1, k_2, ..., k_{n-1})$ such that:

$$K_j \le k_j \le K_j^{'},\ j = 1, 2, ..., n-1 \quad (10)$$

This system of inequalities presents geometrically the (*n*-1)-dimensional rectangular parallelepiped, whose vertices have the coordinates equal to the integer theoretical nesting factors K_j or $K_j^{'}$; thus, these vertices correspond to the Beckmann-McPherson central place models. The actual central place hierarchy (19) corresponds to the inner point of this polyhedron.

Let us introduce the slack variables, presenting the deflection of some central place hierarchy from the theoretical one on each hierarchical level *j*:

$$y_j = k_j - K_j \ge 0; \quad z_j = K_j^{'} - k_j \ge 0, \quad j = 1, 2, ..., n-1 \quad (11)$$

Then each admissible central place hierarchy $k = (k_1, k_2, ..., k_{n-1})$ can be presented as a three-row matrix with non-negative components:

$$X = \begin{bmatrix} k_1 & k_2 & \dots & k_{n-1} \\ y_1 & y_2 & \dots & y_{n-1} \\ z_1 & z_2 & \dots & z_{n-1} \end{bmatrix} \qquad (12)$$

and the actual central place hierarchy corresponds to the matrix

$$X_0 = \begin{bmatrix} k_1^0 & k_2^0 & \dots & k_{n-1}^0 \\ k_1^0 - K_1 & k_2^0 - K_2 & \dots & k_{n-1}^0 - K_{n-1} \\ K_1^{'} - k_1^0 & K_2^{'} - k_2^0 & \dots & K_{n-1}^{'} - k_{n-1}^0 \end{bmatrix} \qquad (13)$$

7.3. Decomposition of an Actual Central Place Hierarchy

According to the superposition principle (see Sonis, 1970, 1980, 1982b, 1985), the hierarchical analysis of an actual central place system represented by the non-negative matrix X_0 is reduced to the decomposition of this matrix into the weighted sum of matrices $X_1, X_2, ..., X_{r+1}$:

$$X_0 = p_1 X_1 + p_2 X_2 + ... + p_{r+1} X_{r+1,} \quad r \le n \qquad (14)$$

where each matrix X_i represents the extreme state of the central place system, corresponding to some Beckmann-McPherson model and the weights p_i have the property:

$$p_1 + p_2 + ... + p_{r+1} = 1;\ 0 \le p_i \le 1;\ r \le n \qquad (15)$$

If we take into consideration only the first row of each matrix in the decomposition (14), we obtain the decomposition of the actual central place hierarchy $k_0 = (k_1^0, k_2^0, ..., k_{n-1}^0)$ into the convex combination of the Beckmann-McPherson central place hierarchies k_i with the same weights p_i :

$$k_0 = p_1 k_1 + p_2 k_2 + ... + p_{r+1} k_{r+1} \quad , r \le n \qquad (16)$$

We interpret the decomposition (15, 16) in the following way: in each actual central place system, there is a set of substantially significant tendencies towards the optimal organization of space in the form of Beckmann-McPherson hierarchies. Geometrically, these tendencies define the simplex enclosed into the polyhedron of admissible central place hierarchies whose vertices correspond to the assemblage of the matrices X_i . An actual central place hierarchy X_0 is the center of gravity of this simplex with the weights p_i . It is possible to interpret the weights p_i in a probabilistic form as the frequencies of the partial realization of some combination of the Christaller-Lösch optimization principles in the hierarchical structure of the actual central place system.

7.3.Best Fitting Approximation Procedure and the Algorithm of Decomposition

The best-fitting procedure of this chapter is a simplification of the procedure proposed by

Parr (1978). This procedure will be used for the derivation of the central place hierarchy on each hierarchical level and in this way will be the basis for the construction of the best fitting simplex that contains the actual central place hierarchy matrix X_0 corresponding to the vector $k_0 = (k_1^0, k_2^0, ..., k_{n-1}^0)$ of average nesting factors. The best-fitting procedure is as follows: for each hierarchical level *i*, the segment $K_i \le k_i^0 \le K_i^{'}$ between the theoretical nesting factors $K_i, K_i^{'}$ can be chosen, which includes the average nesting factor k_i^0 . In this way, the first best fitting Beckman-McPherson model $k_1 = (k_1^1, k_2^1, ..., k_{n-1}^1)$ can be constructed with the help of "best fitting" formulae (Sonis, 1985):

$$k_i^1 = \begin{cases} K_i \quad if \quad k_i^0 \le \dfrac{K_i + K_i^{'}}{2} \\ K_i^{'} \quad if \quad k_i^0 > \dfrac{K_i + K_i^{'}}{2} \end{cases} \qquad (17)$$

In this procedure the values $\dfrac{K_i + K_i^{'}}{2}$ define the boundaries of the domain of structural stability of the decomposition (14, 15).

The weight p_1 of the Beckmann-McPherson model X_1 can be found by the requirement to choose the biggest positive $p\,(0<p<1)$ satisfying the condition $X_0 - p\ X_1 \ge 0$, or in the coordinate form:

$$p_1 = \min_i \ \{1, \frac{k_i^0}{k_i^1}, \frac{k_i^0 - K_i}{k_i^1 - K_i}, \frac{K_i^{'} - k_i^0}{K_i^{'} - k_i^1}\} \qquad (18)$$

The place of the components of the matrices X_0 and X_1 , yielding the minimum in (28), defines the hierarchical level on which there exists the strongest interdiction to the extreme tendency represented by the chosen Beckmann-McPherson model X_1 , on the part of other tendencies acting in the actual central place hierarchy.

The residual X_2 , defined by the equality:

$$X_0 - p_1 X_1 = (1 - p_1)\, X_2 \qquad (19)$$

represents the mutual action of other tendencies developing in the central place hierarchy with the weight $1 - p_1$. This may be interpreted geometrically by constructing a straight line that passes the vertex X_1 and the point X_0 of the actual central place hierarchy and crosses the opposite face of the parallelepiped of admissible central place hierarchies at the point X_2 . Moreover, if one hangs the weights p_1 and $1-p_1$ on points X_1 and X_2 then the center of gravity of the segment with end points X_1 and X_2 will coincide with the point X_0 . For study of the residual X_2 , one should apply the "best fitting" procedure to the X_2 , and so forth.

8. Hierarchical analysis of the Christaller original central place system in Munich, Southern Germany.

After the decades of empirical studies, the pure Christaller-Lösch theoretical hierarchies of several hierarchical levels with the same nesting factors have rarely if ever observed. The reason for this is that each actual central place hierarchy is the superposition of various theoretical hierarchies. It is interesting to note that even Christaller's original study of the Munich central place hierarchy confirms the phenomenon of superposition. The Christaller original Munich central place hierarchy (Christaller, 1933; Woldenberg, 1979, Table 5, p. 446.) can be presented with the help of the following vector of market area frequencies $m_0 = (519, 249, 127, 39, 12, 3, 1)$ with the corresponding sequence of average nesting factors $k_0 = (2.0843, 1.9606, 3.2564, 3.25, 4, 3)$. The polyhedron of admissible central place hierarchies includes all matrices of the form (see 12):

$$X = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 = 4 & k_6 = 3 \\ k_1 - 1 & k_2 - 1 & k_3 - 3 & k_4 - 3 & 0 & 0 \\ 3 - k_1 & 3 - k_2 & 4 - k_3 & 4 - k_4 & 0 & 0 \end{bmatrix}$$

The Munich central place hierarchy is represented by a matrix:

$$X_0 = \begin{bmatrix} 2.0843 & 1.9606 & 3.2564 & 3.25 & 4 & 3 \\ 1.0843 & 0.9606 & 0.2564 & 0.25 & 0 & 0 \\ 0.9157 & 1.0394 & 0.7436 & 0.75 & 0 & 0 \end{bmatrix} \qquad (20)$$

The best-fitting approximation of the vector of average nesting factors $k_0 = (2.0843, 1.9606, 3.2564, 3.25, 4, 3)$ has a form $k_1 = (3, 3, 3, 3, 4, 3)$ which generates the Beckmann-McPherson model

$$X_1 = \begin{bmatrix} 3 & 3 & 3 & 3 & 4 & 3 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad (21)$$

The weight p_1 of this Beckmann-McPherson model is equal to (see (18)):

$$p_1 = \min\left(1, \frac{2.0849}{3}, \frac{1.9606}{3}, \frac{1.0843}{2}, \frac{0.9606}{2}, \frac{0.7436}{1}, \frac{0.75}{1}\right) = \frac{0.9606}{2} = 0.4803 \qquad (22)$$

i.e.

$$X_0 = 0.4803 X_1 + 0.5197 X' \qquad (23)$$

where X' is a residual. Thus, the real central place system X_0 includes only 48.03% of the extreme tendency X_1 corresponding to the best fitting Beckmann-McPherson model. The residual X' can be calculated from equation (23). The best fitting procedure applied to this residual will give us the second extreme tendency X_2 and its weight p_2 . Such a procedure can be repeated once more. After 5 steps the final decomposition of the Munich central place hierarchy can be obtained:

$$\begin{aligned} X_0 &= 0.4803X_1 + 0.2633X_2 + 0.1946X_3 + 0.0554X_4 + 0.0064X_5 = \\ &= 0.4803\begin{bmatrix} 3 & 3 & 3 & 3 & 4 & 3 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} + 0.2633\begin{bmatrix} 1 & 1 & 3 & 3 & 4 & 3 \\ 0 & \bullet & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} + \\ &+ 0.1946\begin{bmatrix} 1 & 1 & 4 & 4 & 4 & 3 \\ 0 & \bullet & 1 & 1 & 0 & 0 \\ 2 & 2 & \bullet & 0 & 0 & 0 \end{bmatrix} + 0.0554\begin{bmatrix} 3 & 1 & 4 & 4 & 4 & 3 \\ 2 & \bullet & 1 & 1 & 0 & 0 \\ \bullet & 2 & \bullet & 0 & 0 & 0 \end{bmatrix} + \\ &+ 0.0064\begin{bmatrix} 3 & 1 & 4 & 3 & 4 & 3 \\ 2 & \bullet & 1 & \bullet & 0 & 0 \\ \bullet & 2 & \bullet & 1 & 0 & 0 \end{bmatrix} \end{aligned} \qquad (24)$$

The first row of this matrix equality gives the decomposition of the vector of average nesting factors:

$$\begin{aligned} k_0 &= (2.0843, 1.9606, 3.2564, 3.25, 4, 3) = \\ &= 0.4803k_1 + 0.2633k_2 + 0.1946k_3 + 0.0554k_4 + 0.0064k_5 = \\ &= 0.4803(3,3,3,3,4,3) + \\ &+ 0.2633(1,1,3,3,4,3) + \\ &+ 0.1946(1,1,4,4,4,3) + \\ &+ 0.0554(3,1,4,4,4,3) + \\ &+ 0.0064(3,1,4,3,4,3) \end{aligned} \qquad \textbf{(25)}$$

This decomposition means that the Munich central place hierarchy consists of five extreme tendencies. The first most prominent tendency corresponds to the Beckmann-McPherson model with nesting factors $k_1 = (3,3,3,3,4,3)$. This tendency consists of the economizing of the number of market areas on almost each hierarchical level; only the second hierarchical level corresponds to economizing of transportation routes. This tendency is very closed to a perfect Christaller hierarchy (3,3,3,3,3,3) and maybe, this was a reason for the introduction by Christaller of his market principle. Nevertheless, the weight of this extreme tendency is equal to $p_1 = 0.4803$ only, i.e., it accounts only for 48.03% of the actual central place phenomenon. The second extreme tendency, corresponding to the Beckmann-McPherson model with the vector of nesting factors $k_2 = (1,1,3,3,4,3)$, interdicts the first tendency on three lower hierarchical levels and represents the tendency of merging of these hierarchical levels, since the vector of nesting factors k_2 includes the nesting factors equal to 1.The second extreme tendency accounts for an additional 26.33% of the phenomenon. The third extreme tendency $k_3 = (1,1,4,4,4,3)$ counteracts the first and second tendencies by implying the passage from market principle to the transportation principle on the forth and fifth hierarchical levels. It explains additionally 19.46% of the phenomenon, so first three extreme tendencies together explain 93.82% of the actual central place hierarchy. The forth and fifth extreme tendencies are not so essential, since they explain together only 6.18% of the rest of phenomenon. It is possible to present the cumulative action of the market and transportation optimization principles of all extreme tendencies separately on each hierarchical level, by accounting the weight of nesting factors 3 and 4 on each hierarchical level (see table 1).

Table 1. Hierarchical structure of the original Christaller central place system of Munich, Southern Germany.

Hierarchical level	Average nesting factors	Nesting factors for Beckmann-McPherson models					Structure of hierarchical levels	
j	k_0	k_1	k_2	k_3	k_4	k_5		
7	–	–	–	–	–	–		
6	3	3	3	3	3	3	$3:100\%$	
5	4	4	4	4	4	4	$4:100\%$	
4	3.25	3	3	4	4	3	$3:75\%$	$4:25\%$
3	3.2564	3	3	4	4	4	$3:74.35\%$	$4:25.65\%$
2	1.9606	3	1	1	1	1	$1:51.97\%$	$3:48.03\%$
1	2.0843	3	1	1	3	3	$1:51.32\%$	$3:48.68\%$
Weigths of Beckmann-McPherson models $(\%)$		48.03	26.32	19.34	5.66	0.65		

The last right column of the table 1 presents the structure of all hierarchical levels of Munich central place hierarchy. We may see that the six hierarchical level includes the hexagonal covering corresponding to the market optimization principle; this covering is generated by the point (1, 1,-1) giving the nesting factor 3. On fifth hierarchical level only the transportation principle appears; the corresponding hexagon covering generated by the point whose barycentric coordinates on the corresponding the Möbius plane are (2,0,-1). On the fourth hierarchical levels the market and transportation principles are acting in proportion 75%/25%. The corresponding hexagon covering is generated with the help of two operations: first operation is the convex combination of market and transportation coverings with the weights 0.75 and 0.25: the corresponding point generated this hexagon covering is:

$$0.75\ (1,\ 1,-1) + 0.25\ (2,\ 0,-1) = (1.25,\ 0.75,-1) \qquad (26)$$

corresponding to the nesting factor 3.0625. the second operation is the homothetic transformation of transfer from this nesting factor to the average nesting factor 3.25; the radius of the homothety is $\sqrt{\frac{3.25}{3.0625}} = 1.1262$, the homothetic transformation (36) has a form (1.2878, 07727, -1.0605) with corresponding nesting factor 3.25.

The third hierarchical level has almost the same structure as fourth level.

The second hierarchical level includes the weighted combination of the market principle covering generated by point (1, 1, -1) and the tendency of merging of hierarchical levels generated by point (1, 0, 0):

0.4803 (1, 1, -1) +0.5197 (1, 0, 0) = (1, 0.4803, -0.4803)

with average nesting factor 1.9606.

The first hierarchical level has almost the same structure as second hierarchical level

Thus, the decomposition analysis of the Christaller example of the Munich, Southern Germany central place hierarchy, hints on the origins of appearance of Christaller optimization principles in the Central Place Theory.

9. Structural Stability, Structural Changes and Catastrophes in Central place Hierarchical Dynamics.

The hierarchical dynamics of the Central place systems are the reflection of the socio-economic spatial complication process of the urban/regional system. The hierarchical dynamics represent both the rapid change and locational and functional inertia within the urban system. The major reason of catastrophic hierarchical change in an evolving urban/regional system is the transfer of a few centers from one hierarchical level to another as a result of changes in allocation of individual central place functions within the hierarchy, i.e., in the modification in the functional extent of the level (*cf.* Parr, 1981, pp. 105-108). The complication process is also reflecting the appearance or disappearance of centers as a result of regional growth, decline or regional competition (see Batty and Friedrich, 2000).

Next we will consider the implementation of the principle of structural stability and structural changes into the dynamics of central place hierarchies.

The main questions of the structural stability and structural changes are:

- What types of central place hierarchies are possible? The Central place theory in its new form, presented in this chapter, is giving the possible answer on this question.
- What kind of structural changes are admissible and what types of structures are preserved (at list partially) under these changes?
- How do the transitions from one type of structure to another occur?

The first question immediately points to the gap between the pure theoretical central place models, based on the Lösch economic landscape, and the hierarchical structure of an actual central place system; two other questions underline the fact that existing central place theory is mostly static and tells us little about the complication process of emergence transformations and stability of an urban hierarchy. A vast body of literature expanding the classical central place theory since its initial formation includes only a small part relevant to the current focus on structural changes within the central place hierarchy. The polyhedral catastrophic dynamics of the states of the central place hierarchy represents its comparative statics and may be considered as a necessary step toward the dynamic theory of the central place hierarchy, which is waiting till now its creator. Between the earlier attempts to construct the dynamic Central place theory the simulation efforts of Morrill (1962), White (1977, 1978), Allen and Sanglier, 1979, Camagni *et al.*, 1986 and Diappi *et.al*, 1990, should be mentioned.

Geometrically, three different types of hierarchical changes are possible.

The first type of change is connected with the case of global structural stability, when the average nesting factors are changing slowly, so the polyhedron of admissible central place hierarchies remains the same and the point of actual central place hierarchy is moving within the same simplex. That means that in the decomposition $X_0 = p_1X_1 + p_2X_2 + ... + p_{r+1}X_{r+1,} \;\; r \le n$ the vertices X_i remains the same and only the coefficients p_i are slowly changing preserving the property $p_1 + p_2 + ... + p_{r+1} = 1; 0 \le p_i \le 1$.

The domain of such movement in the polyhedron of admissible central place hierarchies is the domain of global structural stability. In reality, the domain of structural stability is usually very narrow, and small changes in the average nesting factors caused the crossing the boundary of this domain. This implies the exchange in the decomposition (VII.9) of some extreme tendencies with others, and further, allows even the complete change of composition and ranking of the Beckman-McPherson models entering the decomposition.

The second type of hierarchical change is connected with the transfer of the point of an actual central place X_0 from one convex polyhedron of admissible hierarchies to another convex polyhedron, defined by the different Kanzig-Dacey nesting factors. Geometrically this means the crossing the boundary of the initial polyhedron. In this case, the best we can expect is the partial structural stability, i.e., the stable inclusion in the decomposition (II.9) of only a part of previous extreme tendencies. In reality, the case of the partial structural stability is a most expected one.

The third essentially different type of change in the hierarchical structure is the change in the dimension of the polyhedron of admissible hierarchies. This type of the catastrophic change is caused by the change in a number and content of hierarchical levels as a result of a split or merging of hierarchical levels (*cf*. Parr, 1981, pp. 101-110). The split of a level is characterized by the increase in the number of the central places on the same hierarchical level and a differentiation in the functional content of the level. The merging of the levels is connected with the decrease of a degree of functional differentiation and with the appearance of a new tendency corresponding to the Beckman-McPherson model with the nesting factor equal 1 on the same hierarchical level..

10. Further theoretical developments: short reviews.

The existence of the axiomatic theory of Central Places presented in this paper is pointed on the further directions in empirical studies and further directions of theoretical developments.

In the field of empirical studies the use of superposition model supported by easily assessable computer software (*cf.* the analysis of original Christaller Central place system of Southern Germany in chapter 8) will eventually result in the taxonomy of types of evolution of hierarchical central place system. In such a way the gap between the theory of central place system and its empirical justification will be narrowed.

In the field of theoretical studies the axiomatic theory of Central Places is presented a wide range of possibilities of constructing of stylized examples of the development of evolving socio-economic systems in geographical space. Two main applications of this conceptual framework are elaborated

I. Structurally stable optimal (minimal cost) transportation flows in the hierarchical Central Place system.

II. The merger of two major theories in the Regional Science: the classical Input-Output theory of Leontief and the classical Christaller - *Lösch* Central Place theory.

The application I consider the possibilities of the extension of optimal transportation flow in expending urban system. It is quite understandable that the actual central place hierarchy puts strong restrictions on the type of optimal (minimal cost) flows between the central places. In turn, the spatial and temporal stability of the transportation flows may be an essential factor of growth and decline of the individual central place in the hierarchy. Moreover, usually the optimal transportation flow does not cover all linkages of the transportation network between the central places.

The problem of enumeration of all possible extensions of minimal cost transportation flows is purely combinatorial and hence formidable, cumbersome and tedious. Its solution for each given Beckmann-McPherson central place hierarchical model can be found with the help of aggregated schemes for the transportation tables scheme includes one or two arcs, which present the set of possible linkages between the central places. These arcs are presented as lines in the cells of the aggregated scheme (see Sonis, 1982a;1984; 1986;2000, Huff *et al.*,1986)

Application II describes the mutual penetration and unification two central theories in the Regional Science: the classical Input-Output theory of Leontief and the classical Christaller- Lösch Central place theory (see Sonis, Hewings, 2000b.) It is expected that the proposed theoretical methodology will be useful for the analysis of organization of the production economics in geographical space. In this study the triple *UDL*-factorization is applied to the decomposition of the Leontief Inverse for Input-Output System within the Central Place System of the Christaller- Lösch , Bekmann-McPherson type. Such a factorization reflects the process of gradual extension and complication of the Central Place Hierarchy (see Sonis, Hewings, 2000a).

The idea to investigate the Input-Output relationship within the Central Place system is not a new one. The necessity to combine together the hierarchical structure of Central Place system with Input-Output structure of the transaction flows within one unifying framework was stresses in the programmatic book of Walter Isard (1960, p.141). The first systematic treatment of this problem was undertaken by Robison and Miller (1991). They used the rudimentary structure of intercommunity central place system, without paying attention on the fine structure of the central place hierarchy. The complexity of mathematical presentation stops them on the level of a simple two-community two-order sub region level with one dominant central place.

In the study (Sonis, Hewings 2000b) the central place hierarchy and multi-regional input-output analysis are fusing together, and in a result the decomposition of the Leontief Inverse for Input-Output Central Place system reflects the process of complication of the evolving hierarchy of Central places

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